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Joseph Eyang'an Esekon
Murang'a University College

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ANALYTIC SOLUTION OF A NONLINEAR BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

Joseph Esekon[1], Silas Onyango[2], Naftali Omolo-Ongati[3 §]
[1,3]Department of Mathematics and Applied Statistics
Maseno University
P.O. Box 333, Maseno, KENYA
[1]e-mail: esekon2001@yahoo.com
[3]e-mail: omolo_ongati@yahoo.com
[2]Research Services Office
Strathmore University
P. O. Box 59857, City Square Nairobi, 00200, KENYA
e-mail: sonyango@strathmore.edu

Abstract: We study a nonlinear Black-Scholes partial differential equation whose nonlinearity is as a result of a feedback effect. This is an illiquid market effect arising from transaction costs. An analytic solution to the nonlinear Black-Scholes equation via a solitary wave solution is currently unknown. After transforming the equation into a parabolic nonlinear porous medium equation, we find that the assumption of a traveling wave profile to the later equation reduces it to ordinary differential equations. This together with the use of localizing boundary conditions facilitate a twice continuously differentiable nontrivial analytic solution by integrating directly.

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1. Introduction

Two primary assumptions are used in formulating classical arbitrage pricing

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§Correspondence author

theory: frictionless and competitive markets. In a frictionless market, there are no transaction costs and restrictions on trade while in a competitive market, a trader can buy or sell any quantity of a security without changing its price. Relaxing the competitive market assumption can completely change the standard theory. As such, manipulation of the market may become an issue and pricing of an option becomes market structure and trader dependent.

The notion of liquidity risk is introduced on relaxing the assumptions. This risk, roughly speaking, is the additional risk resulting from timing and the size of a trade.

Under market manipulation, the price process of a security can depend on the entire history of the investor's past trades up to the current trade. Eliminating this path-dependent condition rules out market manipulation and allows use of the classical arbitrage pricing theory.

In the classical theory, there is no change in price for any order size, i.e., the trader does not move the market. The price process of a security is independent of the past. This means that a trading strategy has a temporary impact on the price process.

In illiquid markets, the effect of delta hedging is a function of market liquidity. This feedback effect is such that for transaction costs model if ("Gamma") $u_{ss} < 0$, the "volatility", $\sigma\sqrt{1+2\rho s u_{ss}}$, goes up and vice versa. The feedback effect has to be incorporated into the pricing of derivatives implicitly or explicitly.

An analytic solution to the nonlinear Black-Scholes partial differential equation via a solitary wave solution is currently unknown.

The purpose of the paper is to solve analytically the nonlinear Black-Scholes equation arising from transaction costs. This is done by differentiating the equation twice with respect to the spatial variable s. After substitutions and transformations, we get a porous medium equation. Assuming a traveling wave solution to the porous medium equation reduces it to ordinary differential equations (ODEs) and the solution to the nonlinear Black-Scholes equation follows.

This paper is outlined as follows. Section 2 describes the classical option pricing theory. Its modification is in Section 3. The solution to the nonlinear Black-Scholes equation is presented in Section 4. Section 5 concludes the paper.

2. Standard Option Valuation Theory

The results of the linear Black-Scholes equation were obtained by considering an option maturing at time T for a non-dividend-paying stock. In the classical theory, there is no change in price for any order size, i.e., the trader does not move the market. The price process of a security is independent of the past. This means that an investor's trading strategy has a temporary impact on the price process.

A *call (put)* option is a contract where at a prescribed time in future, known as the *expiry date* T, the holder of the option may buy (sell) a prescribed asset, known as the *underlying asset* s, for a prescribed amount, known as the *exercise (strike) price* K. The opposite party, has the obligation to sell (buy) the asset if the holder chooses to buy (sell) it. An option's value is therefore a function of various parameters in the contract, such as the time to expiry T and strike price K. It also depends on the asset's properties, such as its drift μ and volatility σ, its current market price s_t and time t, and the risk-free interest rate r. The option's value can therefore be written as $u(t, s; \sigma, \mu; K, T; r)$. The following assumptions are used for modeling the financial market:

i. The underlying asset s follows a geometric Brownian motion;

ii. The drift, volatility and the risk-free interest rate are constant for $0 \leq t \leq T$. No dividends are paid in that period;

iii. The market is frictionless, hence there are no transaction costs, lending and borrowing rates of interest are equal, all parties can access any information, and all credits and securities are available in any size at any time. Moreover, the price cannot be influenced by an individual trading, i.e., the market is *competitive*; and

iv. There are no arbitrage opportunities.

The market is said to be *complete* under these assumptions. This means that any asset and any derivative can be hedged or replicated with other assets' portfolio in the market.

The first assumption means that

$$ds_t = \mu s_t dt + \sigma s_t dW_t, \quad \mu > 0, \tag{2.1}$$

where W_t is a standard Brownian motion (or Wiener process). This *linear price trajectory* is called the *Merton-Black-Scholes model.*

We now let Π_t be a portfolio's value of one long option position and a short position in some quantity Δ, delta, of the underlying asset:

$$\Pi_t = u(t, s) - \Delta s. \tag{2.2}$$

From Itô's Lemma we have

$$du = (u_t + \frac{1}{2}\sigma^2 s^2 u_{ss})dt + u_s ds,$$

where $u_t = \frac{\partial u}{\partial t}$, $u_s = \frac{\partial u}{\partial s}$ and $u_{ss} = \frac{\partial^2 u}{\partial s^2}$. Hence, the portfolio changes by

$$d\Pi_t = (u_t + \frac{1}{2}\sigma^2 s^2 u_{ss})dt + (u_s - \Delta)ds \tag{2.3}$$

as Δ is constant during the time step dt.

The risk in our portfolio is the random terms. We can reduce or even eliminate the risk by carefully choosing Δ. From (2.3) the random terms are

$$(u_s - \Delta)ds.$$

We can *delta hedge* by choosing

$$\Delta = u_s. \tag{2.4}$$

This leaves us with a portfolio whose value changes by the amount

$$d\Pi_t = (u_t + \frac{1}{2}\sigma^2 s^2 u_{ss})dt. \tag{2.5}$$

The change is completely riskless. This means that

$$d\Pi_t = r\Pi_t dt, \quad \Pi_0 = 1, \tag{2.6}$$

where $r > 0$ is a continuously compounded interest rate. The security Π is said to be "risk-free" as its dynamics do not have stochastic components. Integrating (2.6) gives

$$\Pi_t = e^{rt}.$$

This is an example of the *no-arbitrage* principle.

Substitute (2.4) into (2.2) to get

$$\Pi_t = u - su_s,$$

then plug into the right hand side of (2.6) to get

$$d\Pi_t = r(u - su_s)dt. \tag{2.7}$$

Comparing equations (2.5) and (2.7) we get

$$(u_t + \frac{1}{2}\sigma^2 s^2 u_{ss})dt = r(u - su_s)dt.$$

Divide through by dt. Rearranging gives the *Black-Scholes equation*

$$u_t + \frac{1}{2}\sigma^2 s^2 u_{ss} + rsu_s - ru = 0 \quad \text{in} \quad \mathbb{R} \times [0, T]. \tag{2.8}$$

To specify values of the derivative at the boundaries where possible values of s and t lie, we use boundary conditions. For a European call option, the boundary conditions are

i. $u(t,0) = 0$ for $0 \leq t \leq T$,

ii. $u(t,s) \sim s - Ke^{-r(T-t)}$ as $s \to \infty$.

The *pay-off function* is given by

$$u(T,s) = (s_T - K)^+ = \max\{s_T - K, 0\} \qquad \text{for} \quad 0 \leq s$$

since it can only be exercised if $s_T > K$. As $s \to \infty$, the option is likely to be exercised since s will exceed K. Since European options may only be exercised on expiry, as maturity approaches, this means that $s - Ke^{-r(T-t)} \approx s - K = u(t,s)$. Hence, the second condition has to be understood as

$$\lim_{s\to\infty} \frac{u(t,s)}{s - Ke^{-r(T-t)}} = 1$$

uniformly for $0 \leq t \leq T$. Since (2.8) is independent of μ, we can choose to work in a risk-neutral world to value derivatives. At maturity, the expected value is $\hat{E}(\max\{s_T - K, 0\})$. Hence, a call option's value becomes

$$c(t,s) = e^{-r(T-t)}\hat{E}\left(\max\{s_T - K, 0\}\right), \quad 0 \leq t \leq T. \tag{2.9}$$

Using Itô's Lemma, the stock price in equation (2.1) is given by

$$s_t = s_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}. \tag{2.10}$$

From (2.10) and the relation $\hat{E}(s_t) = s_0 e^{rt}$, equation (2.9) yields

$$c(t,s) = s_0 N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(s_0/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\ln(s_0/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

The cumulative distribution function for the standard normal distribution is given by

$$N(d_i) = P(Z \leq d_i) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{x^2}{2}}\, dx, \qquad d_i \in \mathbb{R}, \quad i = 1 \text{ or } 2.$$

Reciprocally, for a put option, the terminal condition is given by

$$u(T,s) = (K - s_T)^+ = \max\{K - s_T, 0\} \quad \text{for} \qquad 0 \leq s$$

as it can only be exercised if $K > s_T$. Its boundary conditions are

i. $u(t,0) = Ke^{-r(T-t)}$ for $0 \leq t \leq T$,

ii. $u(t,s) \to 0$ as $s \to \infty$.

The expected value is $\hat{E}[(K - s_T)^+]$ when $t = T$. The put price is

$$p(t,s) = Ke^{-r(T-t)}N(-d_2) - s_0 N(-d_1).$$

3. A Modified Option Valuation Model

Nonlinearities in diffusion models can arise from source terms, insect dispersal, heat conduction and illiquid market effects.

In this work, we will consider the (quadratic) transaction-cost model for modeling illiquid markets. Two assets are used in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time t is $B_t \equiv 1$, and a stock (risky and illiquid asset). The bond's market is assumed to be liquid (or perfectly elastic), see [4].

Cetin et al (see [4], [5]) have put forward the predominant model in the transaction-Cost Model where a fundamental stock price process s_t^0 follows the dynamics

$$ds_t^0 = \mu s_t^0 dt + \sigma s_t^0 dW_t, \qquad t \in [0,T].$$

When trading α shares, the transaction price to be paid by the investor at time t for his purchase/sale is

$$\overline{s}_t(\alpha) = e^{\rho\alpha} s_t^0, \quad \alpha \in \mathbb{R},$$

where ρ is a liquidity parameter with $0 \leq \rho < 1$. A bid-ask-spread with size depending on α is essentially modeled by the transaction price. For a Markovian trading strategy (a strategy of the form $\Phi_t = \phi(t, s_t^0)$) for a smooth function $\phi = u_s$ which is the *hedge ratio*, we have $\phi_s = u_{ss}$.

If the stock and bond positions are Φ_t and β_t respectively where Φ_t is a semimartingale, then the paper value $V_t^M = \Phi_t s_t^0 + \beta_t$. The change in the quadratic variation

$$[\Phi]_t = \int_0^t \left(\phi_s(\tau, s_\tau^0)\sigma s_\tau^0\right)^2 d\tau$$

is $d[\Phi]_t = \left(u_{ss}(t,s_t^0)\sigma s_t^0\right)^2 dt$. Applying Itô formula to $u(t, s_t^0)$ gives

$$du(t,s_t^0) = u_s(t,s_t^0)ds_t^0 + \left(u_t(t,s_t^0) + \frac{1}{2}\sigma^2 (s_t^0)^2 u_{ss}(t,s_t^0)\right) dt. \tag{3.1}$$

In the limit, the wealth dynamics of a self-financing strategy is

$$dV_t^M = \Phi_t ds_t^0 - \rho s_t^0 d[\Phi]_t. \tag{3.2}$$

Since $V_t^M = u(t, s_t^0)$, substitute $d[\Phi]_t$ into (3.2) and apply the uniqueness of semi-martingale decompositions. Equating the deterministic components of the resulting equation and equation (3.1) gives

$$u_t + \frac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0, \qquad u(s_T^0, T) = h(s_T^0) \tag{3.3}$$

where $h(s_T^0)$ is a terminal claim whose hedge cost $u(s_t^0, t)$ is the solution to (3.3). The magnitude of the feedback effect is determined by ρs. Large ρ implies a big market-impact of hedging. If $\rho \to 0$ or no hedging demand, the asset's price follows the standard Black-Scholes model with constant volatility σ.

4. Solution to the Nonlinear Black-Scholes Equation

Theorem 4.1. *If $V(x,t)$ is any positive solution to the nonlinear porous medium equation $V_t + \left(D(V)V_x + \frac{\sigma^2}{4}V^2\right)_x = 0$ in $\mathbb{R} \times (0, \infty)$, then*

$$u(s,t) = \frac{1}{\rho}\left(-\sqrt{s}e^{\frac{ct+\delta}{2}} + s(1 - \ln s)(\frac{1}{4} - \frac{c}{\sigma^2}) + st\left(\frac{\sigma^2}{16} - \frac{c^2}{\sigma^2}\right) - \frac{\sigma^2}{16c}e^{ct+\delta}\right)$$

solves the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 s^2 u_{ss}(1 + 2\rho s u_{ss}) = 0$ for $s \in \mathbb{R}$, $t > 0$, $D(V) = \frac{\sigma^2}{2}V$ and for each $\delta \in \mathbb{R}, c > 0$, $\sigma > 0$ and $1 > \rho > 0$.

Proof. Since the dynamical process (3.3) is first order in t, its solutions are expected to be uniquely prescribed by their initial values

$$u(s, 0) = f(s), \quad -\infty < s < \infty.$$

Differentiate (3.3) twice with respect to s and set $u_{ss} = w$ to get

$$w_t + \frac{\sigma^2 s^2}{2}(1 + 4\rho s w)w_{ss} + 2\rho\sigma^2 s^3 w_s^2 + 2\sigma^2 s(1 + 6\rho s w)w_s + \sigma^2(1 + 6\rho s w)w = 0.$$

Applying the transformations $w = \frac{\upsilon}{\rho s}$ and $x = \ln s$ to the reaction-advection-diffusion equation gives

$$\upsilon_t + \frac{\sigma^2}{2}(1 + 4\upsilon)\upsilon_{xx} + 2\sigma^2\upsilon_x^2 + \frac{\sigma^2}{2}(1 + 4\upsilon)\upsilon_x = 0. \tag{4.1}$$

If we let $\upsilon = \frac{V-1}{4}$ we get $\upsilon_t = \frac{V_t}{4}$, $\upsilon_x = \frac{V_x}{4}$, and $\upsilon_{xx} = \frac{V_{xx}}{4}$. Substituting these expressions into equation (4.1) gives

$$V_t + \frac{\sigma^2}{2}\left(VV_{xx} + V_x^2 + VV_x\right) = 0. \tag{4.2}$$

This is a homogeneous second order nonlinear parabolic PDE of degree one. From this advection-diffusion equation, the Fick's law takes the form [8]

$$\phi(V) = D(V)V_x + \frac{\sigma^2}{4}V^2 \tag{4.3}$$

where $\phi = \phi(V)$ is the flux. Substitute (4.3) into (4.2) to get

$$V_t + \phi(V)_x = 0, \quad c(V) = \phi'(V). \tag{4.4}$$

This nonlinear hyperbolic equation is the conservation law. We can write it as

$$V_t + \left(D(V)V_x + \frac{\sigma^2}{4}V^2\right)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty). \tag{4.5}$$

Assume that the diffusion coefficient D is a power function [8], or

$$D(V) = D_0\left(\frac{V}{V_0}\right)^n \quad \text{for} \quad D_0, \quad V_0 \quad \text{constants} \quad \text{and} \quad n > 0. \tag{4.6}$$

Equation (4.5) together with the constitutive assumption (4.6) is what we call the *porous medium equation*. The equation governs porous flows through a porous domain [8]. For our case, $n = 1$. Expanding (4.5) we get

$$V_t + D(V)V_{xx} + \left(D'(V)V_x + \frac{\sigma^2}{2}V\right)V_x = 0 \tag{4.7}$$

where $D(V)V_{xx}$ is a nonlinear Fickian diffusion term. We recover from the *variable diffusion constant* a nonlinear advection term [8]

$$\left(D'(V)V_x + \frac{\sigma^2}{2}V\right)V_x.$$

This implies a propagation signal whose speed is (see [8])

$$D'(V)V_x + \frac{\sigma^2}{2}V.$$

The term *advection* (or *convection* or *transport*) refers to the physical property's horizontal movement (e.g. the horizontal movement of a density wave), see [8].

Comparing the terms in equations (4.2) and (4.7) we conclude that $D(V) = \frac{\sigma^2}{2}V$. Therefore $D'(V) = \frac{\sigma^2}{2}$. Hence from (4.5) we get the equation

$$V_t + \frac{\sigma^2}{2}(VV_x + \frac{1}{2}V^2)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

We now look for a twice continuously differentiable solution of (4.5) on $\mathbb{R}$.

Proposition 4.2. *If $\nu(\xi)$ is a twice continuously differentiable function,* *and x and t are the spatial and time variables respectively, there exists a traveling wave solution to equation (4.5) of the form*

$$V(x,t) = \nu(x-ct) = \nu(\xi) \quad \text{where} \quad \xi = x - ct \tag{4.8}$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$ and $D(V) = \frac{\sigma^2}{2}V$ such that $V(x,t)$ is a traveling wave *of permanent form which translates to the right with constant speed $c > 0$.*

Proof. $V(x,t)$ is interpreted as the strength of the signal. Equation (4.8) is a bounded solution for the (*signal*) *wave profile* at time t. When the conditions $\nu_1 > 0$ at $\xi \to +\infty$ and $\nu_2 > 0$ at $\xi \to -\infty$ are added to the equation, the traveling wave solution is called a *wavefront solution*. The wavefront solution is termed as a *pulse* if V approaches the same constant values at both plus and minus infinity. Since the initial signal $V(x,0) = \nu(x)$, the profile at time t is represented by $\nu(x-ct)$, which is an initial profile translated to the right ct spatial, see units [8]. The constant c represents the wave speed for a wave propagating undistorted along the characteristics $x - ct =$ constant in spacetime [8]. We interpret the variable $\xi = x - ct$ as a moving coordinate. By the chain rule:

$$V_t = -c\nu'(\xi), \quad V_x = \nu'(\xi), \quad \text{and} \quad V_{xx} = \nu''(\xi),$$

where the prime denotes $\frac{d}{d\xi}$. Substituting these expressions into (4.5), we conclude that $\nu(\xi)$ must satisfy the nonlinear second order ODE

$$-c\nu' + D\nu'' + D'(\nu')^2 + \frac{\sigma^2}{2}\nu\nu' = 0 \tag{4.9}$$

and hence V solves (4.5).

Assume also that the traveling wave is *localized*. This means that at large distances, the solution together with its derivatives are small, or

$$\lim_{x\longrightarrow\pm\infty} V(x,t) = \lim_{x\longrightarrow\pm\infty} V_x(x,t) = \lim_{x\longrightarrow\pm\infty} V_{xx}(x,t) = 0.$$

In this case the function V with the form (4.8) is referred to as a *solitary* wave, see [6]. We now impose the localizing boundary conditions

$$\lim_{\xi\longrightarrow\pm\infty} \nu(\xi) = \lim_{\xi\longrightarrow\pm\infty} \nu'(\xi) = \lim_{\xi\longrightarrow\pm\infty} \nu''(\xi) = 0. \tag{4.10}$$

For a special case $D = 0$, (4.5) reduces to the hyperbolic equation

$$V_t + \frac{\sigma^2}{2}VV_x = 0.$$

Comparing this equation to the basic conservation law (4.4) we get

$$V_t + \phi(V)_x = V_t + \phi'(V)V_x = V_t + c(V)V_x = V_t + \frac{dx}{dt}V_x = \frac{dV}{dt} = 0$$

or $V =$ constant. This means that $c = \frac{\sigma^2}{2}V$ and that V is constant on the characteristic curves. These curves are straight lines since

$$\frac{d^2x}{dt^2} = \frac{dc(V)}{dt} = c'(V)\frac{dV}{dt} = 0.$$

Equation (4.9) can now be solved in closed-form. First write it as

$$\frac{d}{d\xi}(D\nu') + \frac{d}{d\xi}(\frac{\sigma^2}{4}\nu^2 - c\nu) = 0.$$

Integrating with respect to ξ we get the standard form (see [8])

$$\nu' = D^{-1}(c\nu - \frac{\sigma^2}{4}\nu^2 + k) \tag{4.11}$$

where k is a constant of integration. From the localizing boundary conditions (4.10), $k = 0$. Equation (4.11) reduces to

$$0 = c\nu_1 - \frac{\sigma^2}{4}\nu_1^2 = c\nu_2 - \frac{\sigma^2}{4}\nu_2^2$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are the two states of the signal at infinity. Therefore

$$c = \frac{\sigma^2}{4}(\nu_1 + \nu_2) = \frac{\sigma^2}{2}\frac{(\nu_1 + \nu_2)}{2}.$$

Hence, averaging the two known states at infinity yields the wave speed (see [8]). □

Since $D(V) = \frac{\sigma^2}{2}V$, simplifying (4.11) further we conclude that $\nu(\xi)$ satisfies the first order linear autonomous ODE

$$2\frac{d\nu}{d\xi} = \frac{4c}{\sigma^2} - \nu.$$

Rearranging the equation and integrating with respect to ξ gives

$$\nu(\xi) = e^{\frac{\delta - \xi}{2}} + \frac{4c}{\sigma^2}$$

where δ is a constant of integration and hence the solution to (4.5) is

$$V(x,t) = e^{\frac{\delta-(x-ct)}{2}} + \frac{4c}{\sigma^2}.$$

Substituting $\upsilon = \frac{V-1}{4}$ the solution to (4.1) becomes

$$\upsilon(x,t) = \frac{1}{4}e^{\frac{\delta-(x-ct)}{2}} + \frac{c}{\sigma^2} - \frac{1}{4}. \tag{4.12}$$

Substituting the transformations $w = \frac{\upsilon}{\rho s}$ and $x = \ln s$ into (4.12) we get

$$u_{ss} = \frac{1}{\rho}\left(\frac{1}{4s^{3/2}}e^{\frac{ct+\delta}{2}} + \frac{1}{s}\left(\frac{c}{\sigma^2} - \frac{1}{4}\right)\right).$$

Integrating u_{ss} twice with respect to the spatial variable s, we arrive at the solution of the nonlinear Black-Scholes PDE (see Theorem 4.1). □

5. Conclusion

We have studied the hedging of derivatives in illiquid markets. Models where the implementation of a hedging strategy affects the price of the underlying asset have been considered. The principal contribution in this work is the reduction of the nonlinear Black-Scholes PDE into a porous medium equation. Assuming the solution of a forward wave, a classical solution was found for the nonlinear Black-Scholes equation. We have further found out that feedback effects can be modeled by a parabolic equation via a temporal *solitary wave solution*. The *soliton* is a localized bounded traveling wave solution. This was used to get the solution to the nonlinear Black-Scholes equation which can be applied in pricing a European *call* option at time $t > 0$. If we consider the asymptotic for $s \to \infty$, in the linear case a call option's price satisfies $u(s,t) \to$ constant s. In the nonlinear case the solution $u(s,t)$ grows faster than in linear as $s \to \infty$ (see the solution in Theorem 4.1). This reflects the fact that option hedging in illiquid markets is more expensive compared to trading in perfectly liquid markets.

In conclusion, further research needs to be done to solve the equation using other boundary conditions. Future work will also involve investigating the nonlinear porous medium equation using phase-plane techniques.

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