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ANALYTIC SOLUTION OF A NONLINEAR BLACK-SCHOLES EQUATION WITH PRICE SLIPPAGE

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Abstract: We study a nonlinear Black-Scholes partial differential equation whose nonlinearity is as a result of transaction cost and a price slippage impact that lead to market illiquidity with feedback effects. After reducing the equation into a second-order nonlinear partial differential equation, we find that the assumption of a traveling wave profile to the second-order equation reduces it further to ordinary differential equations. Solutions to all these transformed equations facilitate an analytic solution to the nonlinear Black-Scholes equation. We finally show that the option is always more volatile compared to the stock when $\frac{1\mp\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2} < \frac{S_0}{S}e^{rt}$.

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Key Words: analytic solution, feedback effects, illiquid markets, transaction cost, price slippage

1. Introduction

Two primary assumptions are used in formulating classical arbitrage pricing theory: *frictionless* and *competitive* markets. There are no transaction costs and restrictions on trade in a frictionless market. In a competitive market, a trader can buy or sell any quantity of a security without changing its price.

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Restrictions on trade are imposed when we have extreme market conditions. In particular, purchases/short sales are not permitted when the market has a surplus/shortage.

The notion of liquidity risk is introduced on relaxing the assumptions above.

The purpose of this paper is to obtain an analytic solution of the nonlinear Black-Scholes equation arising from the combination of transaction cost and a price slippage impact by Bakstein and Howison in [1]. This is done by substitutions and transformations, which give a second-order nonlinear partial differential equation. Assuming a traveling wave solution to the second-order equation reduces it further to ordinary differential equations (ODEs). All these transformed equations are solved to obtain an analytic solution to the nonlinear Black-Scholes equation.

This paper is organized as follows. Section 2 describes the nonlinear Black-Scholes PDE used for modelling illiquid markets with a price slippage impact. The smooth solution to the nonlinear Black-Scholes equation is presented in Section 3. Section 4 concludes the paper.

2. Bakstein and Howison (2003) Equation

In this section, we will consider the continuous-time feedback effects equation for illiquid markets by Bakstein and Howison in [1]. Two assets are used in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$) whose value at time t is $B_t \equiv 1$, and a stock. The stock is assumed to be risky and illiquid while the bond is assumed to be riskless and liquid. This equation (see Theorem 3.1 of [1]) is given by

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + \tfrac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 u_{SS}^3 + rSu_S - ru = 0, \quad (2.1)$$

where S is the price of the stock, $\rho \geq 0$ is a measure of the liquidity of the market, σ is volatility, $u(S,t)$ is the option price and α is a measure of the *price slippage impact* of a trade felt by all participants of a market (see Theorem 3.1 of [1]).

For instance the terminal condition for a European call option is given by

$$h(S_T) = u(S,T) = \max\{S - K, 0\} \quad \text{for} \quad S \geq 0,$$

where $K > 0$ is the striking price and $h(S_T)$ is a terminal claim whose hedge cost, $u(S_t, t)$, is the solution to (2.1). The boundary conditions for the option are as follows:

$$u(0,t) = 0 \quad \text{for} \quad 0 \leq t \leq T,$$

$$u(S,t) \sim S - Ke^{-r(T-t)} \quad \text{as} \quad S \to \infty.$$

We take the last condition to mean that

$$\lim_{S\to\infty} \frac{u(S,t)}{S-Ke^{-r(T-t)}} = 1$$

uniformly for $0 \le t \le T$ with the constraint $u(S,t) \ge 0$.

Liquidity in (2.1) has been defined through a combination of transaction cost and a price slippage impact. Due to ρ, bid-ask spreads dominate the price elasticity effect. When $\alpha = 1$, this corresponds to no slippage and equation (2.1) reduces to the PDE given by

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + rSu_S - ru = 0. \tag{2.2}$$

The solution to equation (2.2) is found in Theorem 3.0.2 of [3] for $r > 0$, and in Theorem 3.2 of [4] and Theorem 4.1 of [5] for $r = 0$.

The magnitude of the market impact is determined by ρS. Large ρ implies a big market impact of hedging. If $\rho = 0$, the asset's price in equation (2.1) follows the standard Black-Scholes model in [2] with constant volatility σ.

3. Smooth Solution to the Bakstein and Howison (2003) Equation

Lemma 3.1. *If $\nu(\xi)$ is a twice continuously differentiable function, and x and t are the spatial and time variables respectively, then there exists a traveling wave solution to the equation,*

$$V_t + \tfrac{1}{2}\sigma^2(V_{xx} + V_x)\left(1 + 2(V_{xx} + V_x)\right) + \tfrac{1}{2}(1-\alpha)^2\sigma^2(V_{xx} + V_x)^3 + rV_x = 0 \tag{3.1}$$

in $\mathbb{R} \times (0,\infty)$ of the form

$$V(x,t) = \nu(\xi) \quad \text{where} \quad \xi = x - ct, \quad \xi \in \mathbb{R} \tag{3.2}$$

for $0 \le \alpha < 1$, $1 < \alpha \le 2$, $r, \sigma, t > 0$ and $x \in \mathbb{R}$ such that $V(x,t)$ is a traveling wave of permanent form which translates to the right with constant speed $c > 0$.

Proof. Applying the chain rule to (3.2) gives

$$V_t = -c\nu'(\xi), \quad V_x = \nu'(\xi), \quad \text{and} \quad V_{xx} = \nu''(\xi),$$

where the prime denotes $\frac{d}{d\xi}$. Substituting these expressions into (3.1), we conclude that $\nu(\xi)$ must satisfy the nonlinear second order ODE

$$-c\nu' + \tfrac{1}{2}\sigma^2(\nu'' + \nu')\left(1 + 2(\nu'' + \nu')\right) + \tfrac{1}{2}(1-\alpha)^2\sigma^2(\nu'' + \nu')^3 + r\nu' = 0 \tag{3.3}$$

in $\mathbb{R}$ and hence $V(x,t)$ solves (3.1).

By setting $c = r$, the equation resulting from (3.3) can now be solved in a closed-form by first writing it as

$$(1-\alpha)^2(\nu'' + \nu')^2 + 2(\nu'' + \nu') + 1 = 0 \quad \text{in} \quad \mathbb{R},$$

where $(1-\alpha)^2 \neq 0$.

The quadratic equation above is solved to get

$$\nu'' + \nu' = \tfrac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, \quad 0 \leq \alpha < 1, \quad 1 < \alpha \leq 2.$$

Upon integration we get the variable separable standard form (see [6])

$$\nu' = e^{\xi_0 - \xi} + \tfrac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}, \quad 0 \leq \alpha < 1, \quad 1 < \alpha \leq 2, \quad \xi_0, \xi \in \mathbb{R},$$

where ξ_0 is a constant of integration. This is the first order linear autonomous and separable ODE whose solution upon integration is given by

$$\nu(\xi) = -e^{\xi_0 - \xi} + \tfrac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\xi + \delta$$

for $0 \leq \alpha < 1$, $1 < \alpha \leq 2$, $\xi_0, \xi \in \mathbb{R}$, where δ is another constant of integration. Applying the initial condition

$$\nu(0) = 0$$

to the equation above gives

$$\delta = e^{\xi_0}.$$

Hence

$$\nu(\xi) = -e^{\xi_0 - \xi} + \tfrac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\xi + e^{\xi_0}$$

for $0 \leq \alpha < 1$, $1 < \alpha \leq 2$, $\xi_0, \xi \in \mathbb{R}$. Substitution gives

$$V(x,t) = -e^{x_0 - (x - rt)} + \tfrac{-1 \pm \sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}(x - rt) + e^{x_0}, \tag{3.4}$$

for $0 \leq \alpha < 1$, $1 < \alpha \leq 2$, $x_0, x \in \mathbb{R}$, $r > 0$, $t \geq 0$ since $\xi_0 = x_0 - c \cdot 0 = x_0$ and $c = r$. $\square$

Theorem 3.2. *If $V(x,t)$ is any positive solution to the nonlinear equation*

$$V_t + \tfrac{1}{2}\sigma^2(V_{xx} + V_x)\left(1 + 2(V_{xx} + V_x)\right) + \tfrac{1}{2}(1-\alpha)^2\sigma^2(V_{xx} + V_x)^3 + rV_x = 0$$

in $\mathbb{R}\times[0,\infty)$ *then*

$$u(S,t)=\frac{1}{\rho}\left[\left(\frac{-1\pm\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}\left\{\ln\left(\frac{S}{K}\right)-rt\right\}+\frac{S_0}{K}\right)S-S_0e^{rt}\right] \tag{3.5}$$

in $\mathbb{R}\times[0,\infty)$ *solves the nonlinear Black-Scholes equation*

$$u_t+\tfrac{1}{2}\sigma^2S^2u_{SS}(1+2\rho Su_{SS})+\tfrac{1}{2}\rho^2(1-\alpha)^2\sigma^2S^4u_{SS}^3+rSu_S-ru=0 \tag{3.6}$$

for $r,K,S,S_0,\sigma,\rho>0$*,* $t\geq 0$*,* $0\leq\alpha<1$*, and* $1<\alpha\leq 2$*, where* S_0 *is the initial stock price.*

Proof. To obtain the solution to equation (3.6) we apply the transformations $x=\ln\left(\frac{S}{K}\right)$ and $V(x,t)=\rho\frac{u(S,t)}{Ke^x}$ to get

$$u_t=\frac{S}{\rho}V_t,\quad u_S=\frac{1}{\rho}(V_x+V),\quad u_{SS}=\frac{1}{\rho S}\left(V_{xx}+V_x\right).$$

Substituting these expressions into (3.6) gives (3.1). Hence, applying the transformations above into (3.4) gives (3.5). □

Remark 3.3. From equation (3.5), it is clear that when $\frac{1\mp\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}<\frac{S_0}{S}e^{rt}$ then $\frac{Su_S}{u}$ is always greater than one as in [2, 3, 4]. This shows that the option is always more volatile than the stock when this inequality holds.

4. Conclusion

We have studied the hedging of derivatives in the presence of transaction cost and a price slippage impact that lead to market illiquidity with feedback effects. Assuming the solution of a forward wave, a classical solution for the nonlinear Black-Scholes equation was found. The solution obtained can be used for pricing a European *call* option at time $t\geq 0$. It is clear that when $\frac{1\mp\sqrt{1-(1-\alpha)^2}}{(1-\alpha)^2}<\frac{S_0}{S}e^{rt}$ this solution supports the comments in [2, 3, 4] that the option is always more volatile than the stock.

In conclusion, further research needs to be done to find out the solution to the equation when $c\neq r$. Another part of our interest is the use of the put-call-parity to obtain the solution of a put option.

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