# Analysing Four Factor Second Order Models Using Response Surface Methodology with Application in Germination of *Melia volkensii*

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**Abstract**

*Second order models are useful in situations where there are curvilinear effects present in the true response function. Such models have real life applications in a wide variety of fields such as agriculture, biology, and business among others. In such cases the problem is twofold. First is to fit a model for the relationship between the dependent variable and the explanatory variables. Second is to find the values of the predictor variables that optimize the response. The objectives here were to fit second order models involving four independent variables as well as to obtain values for the explanatory variables that optimize the dependent variable. Response surface methodology (RSM) is used both to fit the models as well as to analyze the fitted surfaces. The data obtained by simulation were from a four factor rotatable central composite design (CCD). Results included the fitted models and the tests of adequacy of fit for the models. Optimal values for the independent variables were also given. Contour and surface plots are presented to give a pictorial view of the nature of the response surface. As an application a model for the germination of Melia volkensii experiment was fitted and optimal values of temperature, soil pH and chemical concentration obtained. The work in this paper can be directly applied in many instances where an investigator studies the relationship between four predictor variables and a response. With some relevant adjustments this can be extended to any number of explanatory variables.*

**Keywords:** Response surface methodology, Second order model, Optimal; Four factors an Central composite design.

## INTRODUCTION

Response surface methodology (RSM) is a collection of mathematical and statistical techniques that is useful for the modelling and analysis of problems in which a response of interest is influenced by several variables and the objective is to optimize this response (Montgomery, 2005).

The RSM was initially developed and described by Box and Wilson, (1951). Hill and Hunter, (1966) conducted an extensive review of the literature for RSM emphasizing especially on the practical applications of the method. Mead and Pike, (1975) examined the state of RSM from the statistician's point of view and investigated the extent to which the methodology is used in applied research with particular emphasis on biometric applications. Myers, Khuri and Carter, (1989) evaluated the use of RSM between 1966 and 1988. Over the years RSM has been applied in a wide variety of fields. Examples of the recent applications include (Madamba, 2002), (Hussain *et al.*, 2011), (Pishgar-Komleh *et al.*, 2012), (Anwar *et al.*, 2012), (Hussain and Uddin, 2012), (Krishnaa *et al.*, 2013) and (Zainal *et al.*, 2013)

This paper focuses on the analysis of four factor second order models using RSM. In section 2 second order models are discussed and some of its applications on real life problems. Section 3 concentrates on the procedures of analysing four factor second order models using RSM. Some simulation results and experimental results are presented in section 4. Finally, in Section 5 some conclusions and recommendations are made on the use of four factor second order models.

### Second Order Models

The general approach of the response surface methodology is to use first order models to move to the optimum region and then higher order models to explore this region. When the first order model is found to be inadequate, a second degree model should be fitted and appropriate analysis performed on the fitted model.
The second order response surface is of the form:

$$y = \beta_0 + \sum_{i=1}^{p} \beta_i x_i + \sum_{i=1}^{p} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \varepsilon \quad (1)$$

For p=4, the model becomes

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + \beta_{44} x_4^2 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{14} x_1 x_4 + \beta_{23} x_2 x_3 + \beta_{24} x_2 x_4 + \beta_{34} x_3 x_4 + \varepsilon \quad (2)$$

There are several designs that are used for second order response models. Examples include the $3^k$ designs, the Box-Behken design and the central composite design developed by (Box and Wilson, 1951). In this paper, the central composite design to study second order models were used.

## Analysis of Four Factor Second Order Models

When the experimenter is relatively close to the optimum, the second order model is an adequate approximation.
The second order fitted model is of the form

$$\hat{y} = b_0 + \sum_{i=1}^{p} b_i x_i + \sum_{i=1}^{p} B_{ii} x_i^2 + \sum_{i<j} \sum B_{ij} x_i x_j \quad (3)$$

In matrix notation this is

$$\hat{y} = b_0 + x'\hat{\boldsymbol{b}} + x'\hat{\boldsymbol{B}}x \quad (4)$$

Where $b_0$, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{B}}$ are estimates of the intercept, linear and second order coefficients respectively. $x' = (x_1, x_2, \dots, x_p)$ , $\hat{\boldsymbol{b}}' = (b_1, b_2, \dots, b_p)$ and $\hat{\boldsymbol{B}}$ is the $p * p$ symmetric matrix

$$\hat{\boldsymbol{B}} = \begin{bmatrix} b_{11} & \frac{b_{12}}{2} & \cdots & \frac{b_{1p}}{2} \\ \frac{b_{12}}{2} & b_{22} & \cdots & \frac{b_{2p}}{2} \\ & & \ddots & \\ \frac{b_{1p}}{2} & & & b_{pp} \end{bmatrix} \quad (5)$$

## Location of the Stationary Point

The stationary point is the one in which the response has an optimum value (maximum or minimum)
From differential calculus, this point is obtained differentiating the dependent variable and equating to zero to obtain the corresponding values of the independent variable. In this case,

$$\hat{y} = b_0 + x'\hat{\boldsymbol{b}} + x'\hat{\boldsymbol{B}}x \quad (6)$$

$$\frac{\partial \hat{y}}{\partial x} = \hat{\boldsymbol{b}} + 2\hat{\boldsymbol{B}}x \quad (7)$$

Setting the derivative equal to 0, one can solve for the stationary point of the system:

$$x_s = -\frac{1}{2}\hat{\boldsymbol{B}}^{-1}\hat{\boldsymbol{b}} \quad (8)$$

The predicted response at the stationary point is:

$$\hat{y}_s = b_0 + \frac{1}{2}x_s'\hat{\boldsymbol{b}} \quad (9)$$

## Canonical Analysis

There are several ways to examine the fitted second order response surface. Initially it is desirable to plot response contours. This is done by setting $\hat{y}$ to some specified value $y_0$ and tracing out contours relating $x_1, x_2, \dots, x_p$.

An alternative procedure is to reduce the equation to its canonical form.
That is done by forming the equation:

$$\hat{y} = \hat{y}_s + \sum_{i=1}^{p} \lambda_i w_i^2 \quad (10)$$

where $\hat{y}_s$ ,the estimated stationary point is the center of the contours and $w_i s$ are a new set of axes called the principal axes. The coefficients $\lambda_i s$ are the eigenvalues of $\hat{B}$ and give the shape of the surface such that:
If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all negative, the stationary point is a point of maximum response.
If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all positive, the stationary point is a point of minimum response.
If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are mixed in sign, the stationary point is a saddle point.

The relative sizes of the eigenvalues also tell a great deal. For example, if most of the eigenvalues are large positive numbers but a few are near zero, then there is a ridge in the graph of the response function. Moving along that ridge will make little difference in the value of the response but might make a big difference in some other aspect of the system, like cost, for example.

The $w_i s$ are obtained as follows: For a matrix M with columns equal to the normalized eigenvectors of $\hat{B}$, then $M'\hat{B}M = \Lambda$ where $\Lambda$ is a diagonal matrix with diagonal elements equal to the eigenvalues of $\hat{B}$.
Let
$z = x - x_s$,$w = M'z$. Now, $\hat{y} = b_0 + x'\hat{\boldsymbol{b}} + x'\hat{\boldsymbol{B}}x$

$$\hat{y} = b_0 + (z + x_s)'\hat{\boldsymbol{b}} + (z + x_s)'\hat{\boldsymbol{B}}(z + x_s) \quad (11)$$

$$\hat{y} = b_0 + z'\hat{\boldsymbol{b}} + x_s{}'\hat{\boldsymbol{b}} + (z'\hat{\boldsymbol{B}} + x_s{}'\hat{\boldsymbol{B}})(z + x_s) \quad (12)$$

$$\hat{y} = b_0 + z'\hat{\boldsymbol{b}} + x_s{}'\hat{\boldsymbol{b}} + z'\hat{\boldsymbol{B}}z + x_s{}'\hat{\boldsymbol{B}}z + z'\hat{\boldsymbol{B}}x_s + x_s{}'\hat{\boldsymbol{B}}x_s \quad (13)$$

$$\hat{y} = \left[b_0 + x_s'\hat{\boldsymbol{b}} + x_s'\hat{\boldsymbol{B}}x_s\right] + z'\hat{\boldsymbol{b}} + z'\hat{\boldsymbol{B}}z + 2x_s'\hat{\boldsymbol{B}}z \quad (14)$$

But $x_s = -\frac{1}{2}\hat{\boldsymbol{B}}^{-1}\hat{\boldsymbol{b}}$. This implies$2x_s'\hat{\boldsymbol{B}}z = -z'\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}^{-1}\hat{\boldsymbol{b}} = -z'\hat{\boldsymbol{b}}$. Therefore:

$$\hat{y} = \hat{y}_s + z'\hat{\boldsymbol{B}}z \quad (15)$$

Changing the coordinate system:

$$\hat{y} = \hat{y}_s + w'M'\hat{\boldsymbol{B}}Mw \quad (16)$$

$$\hat{y} = \hat{y}_s + w'\Lambda w \quad (17)$$

For p = 4,

$$\hat{y} = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + B_{11}x_1^2 + B_{12}x_1x_2 + B_{13}x_1x_3 + B_{14}x_1x_4 + B_{22}x_2^2 + B_{23}x_2x_3 + B_{24}x_2x_4 + B_{33}x_3^2 + B_{34}x_3x_4 + B_{44}x_4^2 \quad (18)$$

Its canonical form equivalent is:

$$\hat{y} = \hat{y}_s + \lambda_1 w_1^2 + \lambda_2 w_2^2 + \lambda_3 w_3^2 + \lambda_4 w_4^2 \quad (19)$$

where $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ are the eigenvalues of

$$\hat{B} = \begin{bmatrix} B_{11} & \frac{B_{12}}{2} & \frac{B_{13}}{2} & \frac{B_{14}}{2} \\ \frac{B_{12}}{2} & B_{22} & \frac{B_{23}}{2} & \frac{B_{24}}{2} \\ \frac{B_{13}}{2} & \frac{B_{32}}{2} & B_{33} & \frac{B_{34}}{2} \\ \frac{B_{14}}{2} & \frac{B_{24}}{2} & \frac{B_{34}}{2} & B_{44} \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \\ x_3 - x_{3s} \\ x_4 - x_{4s} \end{bmatrix} \quad (21)$$

$\begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}'$ being a matrix with columns equal to the normalized eigenvectors of $\hat{B}$ .

## Empirical Study

### Description of the Empirical Study

Data were simulated corresponding to the three possible response surfaces; maximum response, minimum response and saddle response. There were three replicates for each of these cases. The design was a rotatable four factor central composite design. The models used in the simulation were:

Maximum response

$$y = 10 + 0.8x_1 + 0.5x_2 + 0.3x_3 + 0.1x_4 - 2x_1^2 - 1.7x_1x_2 + 1.4x_1x_3 + 1.2x_1x_4 - 1.6x_2^2 + 0.9x_2x_3 + 0.4x_2x_4 - 0.8x_3^2 - 0.5x_3x_4 - 0.6x_4^2 + \varepsilon \quad (22)$$

Minimum response

$$y = 10 + 0.8x_1 - 0.5x_2 + 0.3x_3 + 0.1x_4 + 2x_1^2 - 1.7x_1x_2 + 1.4x_1x_3 - 1.2x_1x_4 + 1.6x_2^2 - 0.9x_2x_3 + 0.4x_2x_4 + 0.8x_3^2 - 0.5x_3x_4 + 0.6x_4^2 + \varepsilon \quad (23)$$

Saddle response

$$y = 10 + 0.8x_1 + 0.5x_2 + 0.3x_3 - 0.1x_4 - 2x_1^2 + 1.7x_1x_2 + 1.4x_1x_3 + 1.2x_1x_4 + 1.6x_2^2 - 0.9x_2x_3 + 0.4x_2x_4 + 0.8x_3^2 + 0.5x_3x_4 - 0.6x_4^2 + \varepsilon \quad (24)$$

For each of the models $\varepsilon \sim N(0,1)$

The simulated data are given below

Table 1: Simulated Data for the Maximum, Minimum and Saddle Response Models

| Design Variables | | | | Maximum | | | Minimum | | | Saddle | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $y_1$ | $y_2$ | $y_3$ | $y_1$ | $y_2$ | $y_3$ | $y_1$ | $y_2$ | $y_3$ |
| -1 | -1 | -1 | -1 | 4 | 4 | 4 | 11 | 11 | 10 | 11 | 12 | 12 |
| 1 | -1 | -1 | -1 | 5 | 5 | 4 | 18 | 18 | 19 | 7 | 7 | 8 |
| -1 | 1 | -1 | -1 | 8 | 9 | 9 | 15 | 16 | 18 | 11 | 10 | 10 |
| 1 | 1 | -1 | -1 | 0 | 1 | 3 | 13 | 11 | 11 | 11 | 10 | 11 |
| -1 | -1 | 1 | -1 | 2 | 2 | 4 | 14 | 15 | 14 | 10 | 11 | 11 |
| 1 | -1 | 1 | -1 | 7 | 7 | 9 | 20 | 20 | 18 | 10 | 10 | 12 |
| -1 | 1 | 1 | -1 | 8 | 7 | 5 | 12 | 10 | 9 | 6 | 7 | 7 |
| 1 | 1 | 1 | -1 | 5 | 5 | 5 | 19 | 18 | 16 | 13 | 14 | 12 |
| -1 | -1 | -1 | 1 | 3 | 3 | 4 | 13 | 12 | 12 | 9 | 10 | 11 |
| 1 | -1 | -1 | 1 | 8 | 8 | 7 | 15 | 15 | 15 | 6 | 6 | 8 |
| -1 | 1 | -1 | 1 | 7 | 6 | 5 | 18 | 18 | 18 | 8 | 8 | 6 |
| 1 | 1 | -1 | 1 | 5 | 6 | 4 | 14 | 16 | 16 | 13 | 13 | 14 |
| -1 | -1 | 1 | 1 | 0 | 2 | 0 | 14 | 13 | 13 | 9 | 9 | 10 |
| 1 | -1 | 1 | 1 | 10 | 11 | 10 | 18 | 16 | 15 | 12 | 12 | 13 |
| -1 | 1 | 1 | 1 | 4 | 2 | 2 | 15 | 17 | 17 | 5 | 5 | 7 |
| 1 | 1 | 1 | 1 | 8 | 6 | 4 | 12 | 14 | 12 | 16 | 15 | 15 |
| -2 | 0 | 0 | 0 | 1 | 1 | 0 | 17 | 17 | 18 | 3 | 4 | 5 |
| 2 | 0 | 0 | 0 | 3 | 3 | 2 | 18 | 20 | 20 | 4 | 3 | 3 |
| 0 | -2 | 0 | 0 | 2 | 1 | 1 | 17 | 17 | 17 | 16 | 15 | 13 |
| 0 | 2 | 0 | 0 | 5 | 4 | 4 | 16 | 16 | 16 | 19 | 20 | 21 |
| 0 | 0 | -2 | 0 | 7 | 7 | 6 | 9 | 8 | 7 | 14 | 13 | 12 |
| 0 | 0 | 2 | 0 | 7 | 7 | 6 | 13 | 12 | 12 | 14 | 14 | 14 |
| 0 | 0 | 0 | -2 | 8 | 7 | 9 | 12 | 12 | 12 | 8 | 8 | 9 |
| 0 | 0 | 0 | 2 | 7 | 9 | 9 | 12 | 11 | 14 | 8 | 9 | 9 |
| 0 | 0 | 0 | 0 | 11 | 11 | 11 | 9 | 7 | 7 | 10 | 10 | 11 |
| 0 | 0 | 0 | 0 | 9 | 9 | 8 | 10 | 8 | 9 | 9 | 8 | 9 |
| 0 | 0 | 0 | 0 | 12 | 13 | 15 | 10 | 9 | 8 | 11 | 11 | 11 |
| 0 | 0 | 0 | 0 | 9 | 9 | 10 | 9 | 10 | 11 | 12 | 12 | 14 |
| 0 | 0 | 0 | 0 | 11 | 11 | 13 | 9 | 10 | 11 | 9 | 9 | 9 |
| 0 | 0 | 0 | 0 | 10 | 9 | 8 | 10 | 9 | 7 | 10 | 10 | 11 |

## RESULTS AND DISCUSSION

### Analysis of the Maximum Response Model

The characteristics of the fitted maximum response models are summarized in tables 2 and 3 below.

Table 2: The Fitted Maximum Response Model

| | Replicate I | | | | Replicate II | | | | Replicate III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Coeff. | Std.Error | t value | p value | Coeff. | Std. Error | t value | p value | Coeff. | Std. Error | t value | p value |
| Intercept | 10.333 | 0.390 | 26.517 | 0.000 | 10.333 | 0.529 | 19.541 | 0.000 | 10.833 | 0.751 | 14.427 | 0.000 |
| $x_1$ | 0.667 | 0.195 | 3.422 | 0.004 | 0.750 | 0.264 | 2.837 | 0.013 | 0.708 | 0.375 | 1.887 | 0.079 |
| $x_2$ | 0.500 | 0.195 | 2.566 | 0.021 | 0.250 | 0.264 | 0.946 | 0.359 | 0.042 | 0.375 | 0.111 | 0.913 |
| $x_3$ | 0.167 | 0.195 | 0.855 | 0.406 | 0.000 | 0.264 | 0.000 | 1.000 | -0.042 | 0.375 | -0.111 | 0.913 |
| $x_4$ | 0.167 | 0.195 | 0.855 | 0.406 | 0.333 | 0.264 | 1.261 | 0.227 | -0.292 | 0.375 | -0.777 | 0.449 |
| $x_1^2$ | -2.042 | 0.182 | -11.202 | 0.000 | -2.021 | 0.247 | -8.171 | 0.000 | -2.406 | 0.351 | -6.851 | 0.000 |
| $x_1x_2$ | -1.875 | 0.239 | -7.857 | 0.000 | -1.625 | 0.324 | -5.018 | 0.000 | -1.438 | 0.460 | -3.126 | 0.007 |
| $x_1x_3$ | 1.250 | 0.239 | 5.238 | 0.000 | 1.125 | 0.324 | 3.474 | 0.003 | 1.313 | 0.460 | 2.854 | 0.012 |
| $x_1x_4$ | 1.375 | 0.239 | 5.762 | 0.000 | 1.375 | 0.324 | 4.246 | 0.001 | 0.938 | 0.460 | 2.039 | 0.060 |
| $x_2^2$ | -1.667 | 0.182 | -9.145 | 0.000 | -1.896 | 0.247 | -7.665 | 0.000 | -2.031 | 0.351 | -5.784 | 0.000 |
| $x_2x_3$ | 0.375 | 0.239 | 1.572 | 0.137 | -0.250 | 0.324 | -0.772 | 0.452 | -0.563 | 0.460 | -1.223 | 0.240 |
| $x_2x_4$ | 0.000 | 0.239 | 0.000 | 1.000 | -0.500 | 0.324 | -1.544 | 0.143 | -0.438 | 0.460 | -0.951 | 0.356 |
| $x_3^2$ | -0.792 | 0.182 | -4.344 | 0.001 | -0.771 | 0.247 | -3.117 | 0.007 | -1.156 | 0.351 | -3.292 | 0.005 |
| $x_3x_4$ | -0.375 | 0.239 | -1.572 | 0.137 | -0.250 | 0.324 | -0.772 | 0.452 | -0.438 | 0.460 | -0.951 | 0.356 |
| $x_4^2$ | -0.667 | 0.182 | -3.658 | 0.002 | -0.521 | 0.247 | -2.106 | 0.052 | -0.406 | 0.351 | -1.157 | 0.265 |
| Multiple $R^2$ | 0.9571 | | | | 0.9233 | | | | 0.8734 | | | |
| Adjusted $R^2$ | 0.9171 | | | | 0.8517 | | | | 0.7553 | | | |
| F statistic | 23.9200 | | | | 12.9000 | | | | 7.3940 | | | |
| P value | 0.0000 | | | | 0.0000 | | | | 0.0002 | | | |

Table 2 and table 3 show that all the three models are significant with p values of 0.0000, 0.0000 and 0.0002

Table 3: Analysis of Variance Table for the Maximum Response Model

| | Source | Sum of Squares | Degrees of Freedom | Mean Square | F value | P value |
|---|---|---|---|---|---|---|
| Replicate I | Model | 305.133 | 14 | 21.795 | 23.920 | 0.000 |
| | First Order | 18.000 | 4 | 4.500 | 4.939 | 0.010 |
| | Two way interaction | 116.000 | 6 | 19.333 | 21.220 | 0.000 |
| | Pure Quadratic | 171.133 | 4 | 42.783 | 46.957 | 0.000 |
| | Residuals | 13.667 | 15 | 0.911 | | |
| | Lack of fit | 6.333 | 10 | 0.633 | 0.432 | 0.879 |
| | Pure error | 7.333 | 5 | 1.467 | | |
| | Total | 318.800 | 29 | | | |
| Replicate II | Model | 303.000 | 14 | 21.643 | 12.898 | 0.000 |
| | First Order | 17.667 | 4 | 4.417 | 2.633 | 0.076 |
| | Two way interaction | 98.750 | 6 | 16.458 | 9.810 | 0.000 |
| | Pure Quadratic | 186.583 | 4 | 46.646 | 27.802 | 0.000 |
| | Residuals | 25.167 | 15 | 1.678 | | |
| | Lack of fit | 11.833 | 10 | 1.183 | 0.444 | 0.872 |
| | Pure error | 13.333 | 5 | 2.667 | | |
| | Total | 328.167 | 29 | | | |
| Replicate III | Model | 350.217 | 14 | 25.016 | 7.394 | 0.000 |
| | First Order | 14.167 | 4 | 3.542 | 1.047 | 0.416 |
| | Two way interaction | 85.875 | 6 | 14.312 | 4.230 | 0.011 |
| | Pure Quadratic | 250.175 | 4 | 62.544 | 18.486 | 0.000 |
| | Residuals | 50.750 | 15 | 3.383 | | |
| | Lack of fit | 11.917 | 10 | 1.192 | 0.153 | 0.994 |
| | Pure error | 38.833 | 5 | 7.767 | | |
| | Total | 400.967 | 29 | | | |

The stationary points, eigen values and canonical equivalent models are depicted in table 4 below:

Table 4: Stationary points and Eigenvalues and for the Maximum Response Model

| | Stationary Points | | | | Eigenvalues | | | |
|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | | | |
| Replicate I | 0.485 | -0.084 | 0.343 | 0.529 | -0.237 | -0.574 | -1.238 | -3.119 |
| Replicate II | 4.629 | -3.051 | 2.697 | 7.247 | -0.028 | -0.582 | -1.701 | -2.898 |
| Replicate III | 0.139 | -0.028 | 0.114 | -0.245 | -0.216 | -0.674 | -2.036 | -3.075 |

Since for all the replicates all the eigenvalues are negative the response surface is a maximum one.
The canonical equivalent forms of the fitted models are respectively for the three replicates;

$$\hat{y} = 10.546 - 0.237w_1^2 - 0.574w_2^2 - 1.238w_3^2 - 3.119w_4^2 \quad (25)$$

$$\hat{y} = 12.894 - 0.028w_1^2 - 0.582w_2^2 - 1.701w_3^2 - 2.898w_4^2 \quad (26)$$

$$\hat{y} = 10.915 - 0.216w_1^2 - 0.674w_2^2 - 2.036w_3^2 - 3.075w_4^2 \quad (27)$$

The nature of these responses can be seen in the response surface plots shown in figure 1 and figure 2 below.

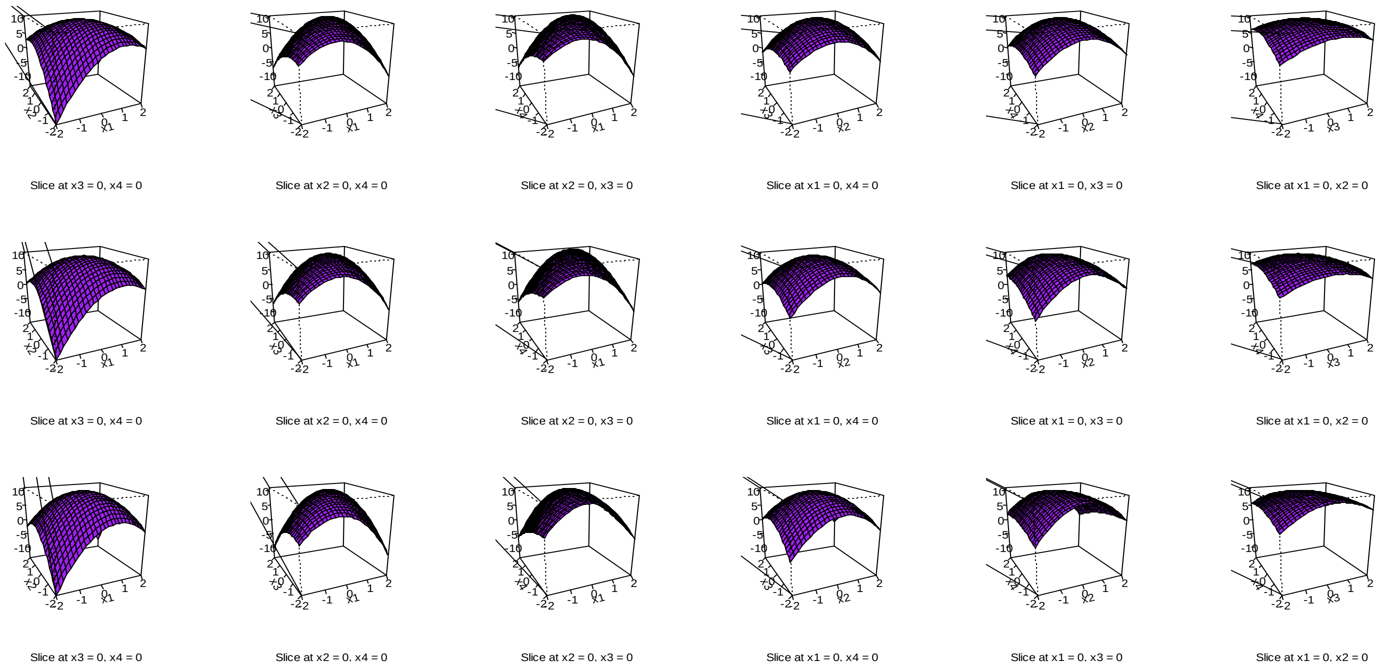

Figure 1: Response Surface Plot for the Maximum Response Model

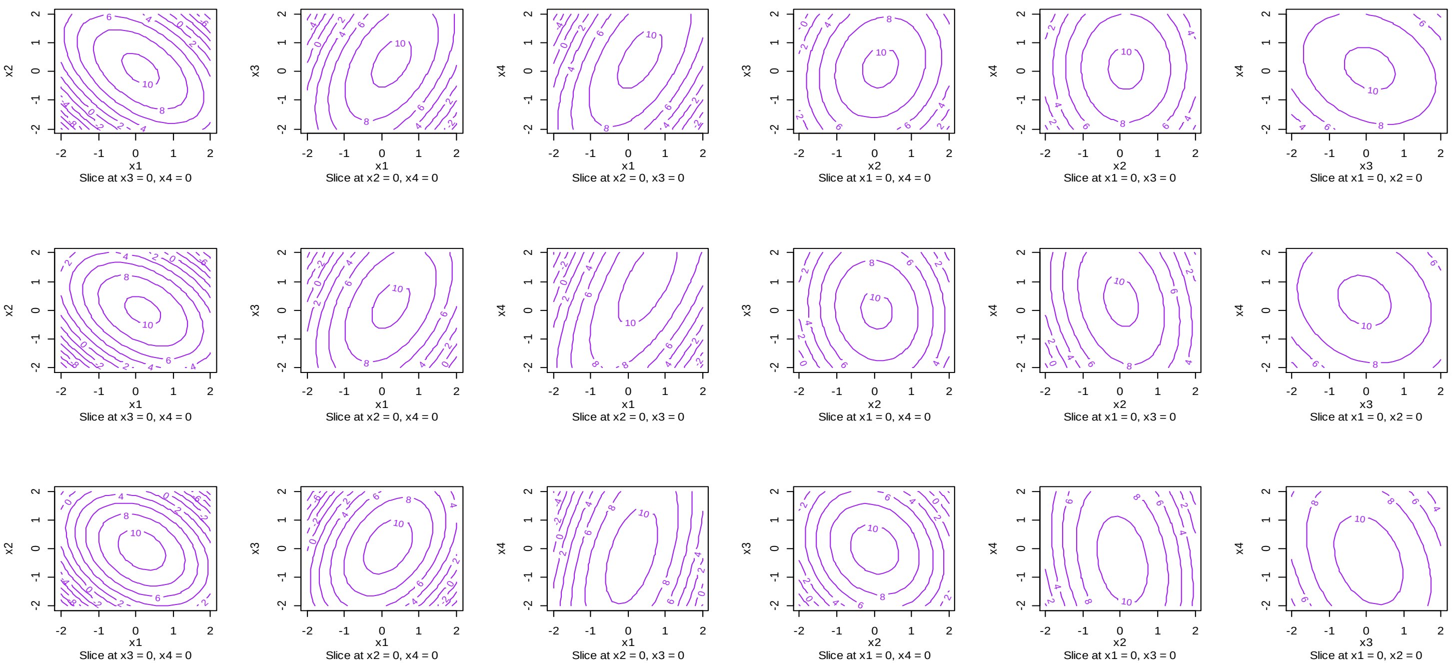


Figure 2: Contour Plot for the Maximum Response Model

**Analysis of the Minimum Response Model**

The characteristics of the fitted minimum response models are summarized in tables 5 and 6.

The tables show that all the three models are significant with p values of 0.0000, 0.0002 and 0.0040.

Table 5: The Fitted Minimum Response Model

| | Replicate I | | | | Replicate II | | | | Replicate III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Coeff. | Std. Error | t value | p value | Coeff. | Std. Error | t value | p value | Coeff. | Std. Error | t value | p value |
| Intercept | 9.500 | 0.570 | 16.664 | 0.000 | 8.833 | 0.760 | 11.630 | 0.000 | 8.833 | 0.986 | 8.963 | 0.000 |
| $x_1$ | 0.792 | 0.285 | 2.777 | 0.014 | 0.917 | 0.380 | 2.414 | 0.029 | 0.625 | 0.493 | 1.268 | 0.224 |
| $x_2$ | -0.292 | 0.285 | -1.023 | 0.322 | -0.083 | 0.380 | -0.219 | 0.829 | -0.042 | 0.493 | -0.085 | 0.934 |
| $x_3$ | 0.625 | 0.285 | 2.193 | 0.045 | 0.583 | 0.380 | 1.536 | 0.145 | 0.208 | 0.493 | 0.423 | 0.678 |
| $x_4$ | -0.125 | 0.285 | -0.439 | 0.667 | 0.000 | 0.380 | 0.000 | 1.000 | 0.292 | 0.493 | 0.592 | 0.563 |
| $x_1^2$ | 2.135 | 0.267 | 8.009 | 0.000 | 2.563 | 0.355 | 7.214 | 0.000 | 2.552 | 0.461 | 5.537 | 0.000 |
| $x_1x_2$ | -1.313 | 0.349 | -3.760 | 0.002 | -1.250 | 0.465 | -2.688 | 0.017 | -1.563 | 0.604 | -2.589 | 0.021 |
| $x_1x_3$ | 0.688 | 0.349 | 1.969 | 0.068 | 0.625 | 0.465 | 1.344 | 0.199 | 0.313 | 0.604 | 0.518 | 0.612 |
| $x_1x_4$ | -1.188 | 0.349 | -3.402 | 0.004 | -0.875 | 0.465 | -1.881 | 0.079 | -0.938 | 0.604 | -1.553 | 0.141 |
| $x_2^2$ | 1.885 | 0.267 | 7.071 | 0.000 | 2.063 | 0.355 | 5.806 | 0.000 | 1.927 | 0.461 | 4.181 | 0.001 |
| $x_2x_3$ | -0.688 | 0.349 | -1.969 | 0.068 | -0.625 | 0.465 | -1.344 | 0.199 | -0.813 | 0.604 | -1.346 | 0.198 |
| $x_2x_4$ | 0.188 | 0.349 | 0.537 | 0.599 | 1.125 | 0.465 | 2.419 | 0.029 | 0.938 | 0.604 | 1.553 | 0.141 |
| $x_3^2$ | 0.510 | 0.267 | 1.914 | 0.075 | 0.438 | 0.355 | 1.232 | 0.237 | 0.177 | 0.461 | 0.384 | 0.706 |
| $x_3x_4$ | -0.563 | 0.349 | -1.611 | 0.128 | -0.500 | 0.465 | -1.075 | 0.299 | -0.188 | 0.604 | -0.311 | 0.760 |
| $x_4^2$ | 0.760 | 0.267 | 2.852 | 0.012 | 0.813 | 0.355 | 2.287 | 0.037 | 1.052 | 0.461 | 2.282 | 0.037 |
| Multiple $R^2$ | 0.9097 | | | | 0.8756 | | | | 0.8010 | | | |
| Adjusted $R^2$ | 0.8254 | | | | 0.7596 | | | | 0.6152 | | | |
| F statistic | 10.7900 | | | | 7.544 | | | | 4.312 | | | |
| P value | 0.0000 | | | | 0.0002 | | | | 0.0040 | | | |

Table 6: Analysis of Variance Table for the Minimum Response Model

| | Source | Sum of Squares | Degrees of Freedom | Mean Square | F value | P value |
|---|---|---|---|---|---|---|
| Replicate I | Model | 294.616 | 14 | 21.044 | 10.792 | 0.000 |
| | First Order | 26.833 | 4 | 6.708 | 3.440 | 0.035 |
| | Two way interaction | 70.875 | 6 | 11.812 | 6.058 | 0.002 |
| | Pure Quadratic | 196.908 | 4 | 49.227 | 25.245 | 0.000 |
| | Residuals | 29.250 | 15 | 1.950 | | |
| | Lack of fit | 27.750 | 10 | 2.775 | 9.250 | 0.012 |
| | Pure error | 1.500 | 5 | 0.300 | | |
| | Total | 323.866 | 29 | | | |
| Replicate II | Model | 365.550 | 14 | 26.111 | 7.544 | 0.000 |
| | First Order | 28.500 | 4 | 7.125 | 2.059 | 0.137 |
| | Two way interaction | 74.000 | 6 | 12.333 | 3.563 | 0.021 |
| | Pure Quadratic | 263.050 | 4 | 65.762 | 19.000 | 0.000 |
| | Residuals | 51.917 | 15 | 3.461 | | |
| | Lack of fit | 45.083 | 10 | 4.508 | 3.299 | 0.100 |
| | Pure error | 6.833 | 5 | 1.367 | | |
| | Total | 417.467 | 29 | | | |
| Replicate III | Model | 351.783 | 14 | 25.127 | 4.311 | 0.004 |
| | First Order | 12.500 | 4 | 3.125 | 0.536 | 0.711 |
| | Two way interaction | 79.875 | 6 | 13.312 | 2.284 | 0.091 |
| | Pure Quadratic | 259.408 | 4 | 64.852 | 11.128 | 0.000 |
| | Residuals | 87.417 | 15 | 5.828 | | |
| | Lack of fit | 70.583 | 10 | 7.058 | 2.097 | 0.214 |
| | Pure error | 16.833 | 5 | 3.367 | | |
| | Total | 439.200 | 29 | | | |

The stationary points and eigenvalues are depicted in table 7:

Table 7: Stationary points and Eigenvalues for the Minimum Response Model

| | Stationary Points | | | | Eigenvalues | | | |
|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | | | |
| Replicate I | -0.190 | -0.107 | -0.735 | -0.325 | 2.932 | 1.447 | 0.632 | 0.281 |
| Replicate II | -0.148 | -0.067 | -0.762 | -0.268 | 3.267 | 1.685 | 0.615 | 0.308 |
| Replicate III | -0.166 | -0.238 | -1.095 | -0.204 | 3.317 | 1.464 | 0.840 | 0.087 |

Since for all the replicates all the eigenvalues were positive the response surface is a minimum one.
The canonical equivalent forms of the fitted models are respectively for the three replicates;

$$\hat{y} = 9.231 + 2.932w_1^2 + 1.447w_2^2 + 0.632w_3^2 + 0.281w_4^2 \quad (28)$$

$$\hat{y} = 8.046 + 3.267w_1^2 + 1.685w_2^2 + 0.615w_3^2 + 0.308w_4^2 \quad (29)$$

$$\hat{y} = 8.143 + 3.317w_1^2 + 1.464w_2^2 + 0.840w_3^2 + 0.087w_4^2 \quad (30)$$

The nature of these responses can be seen in the response surface plots shown in figure 3 and figure 4 below.

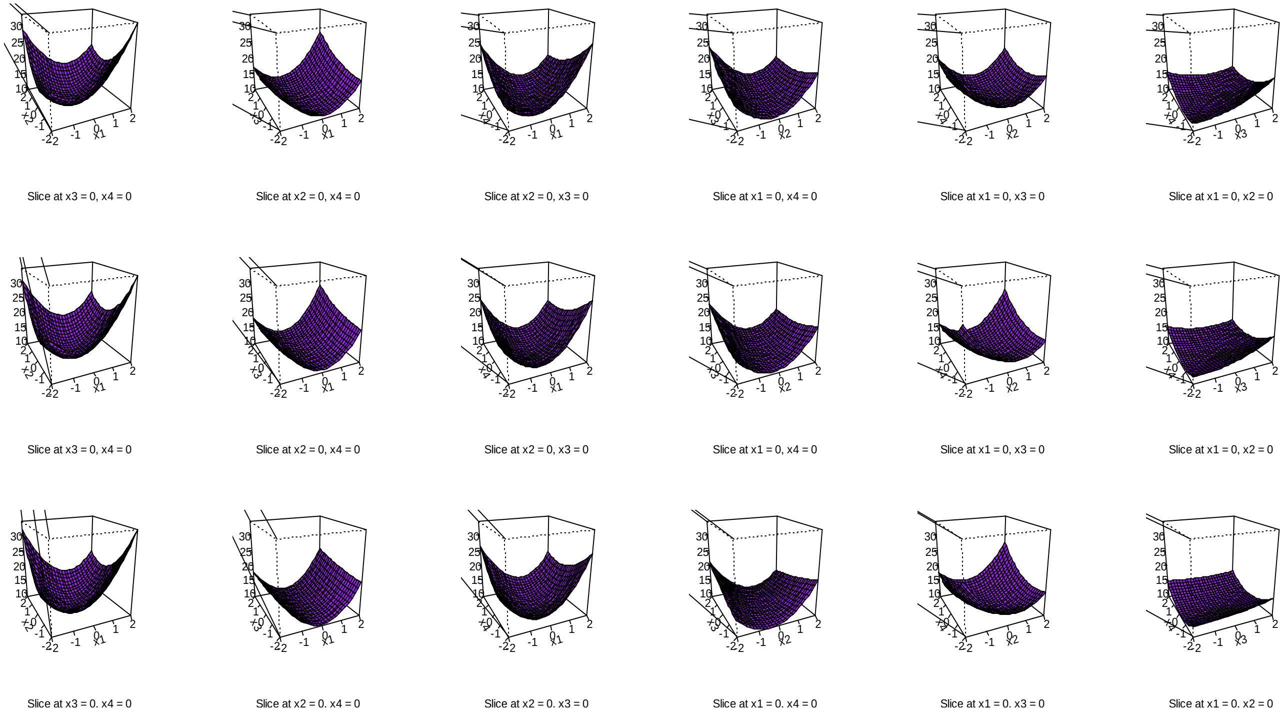


Figure 3: Response Surface Plot for the Minimum Response Model

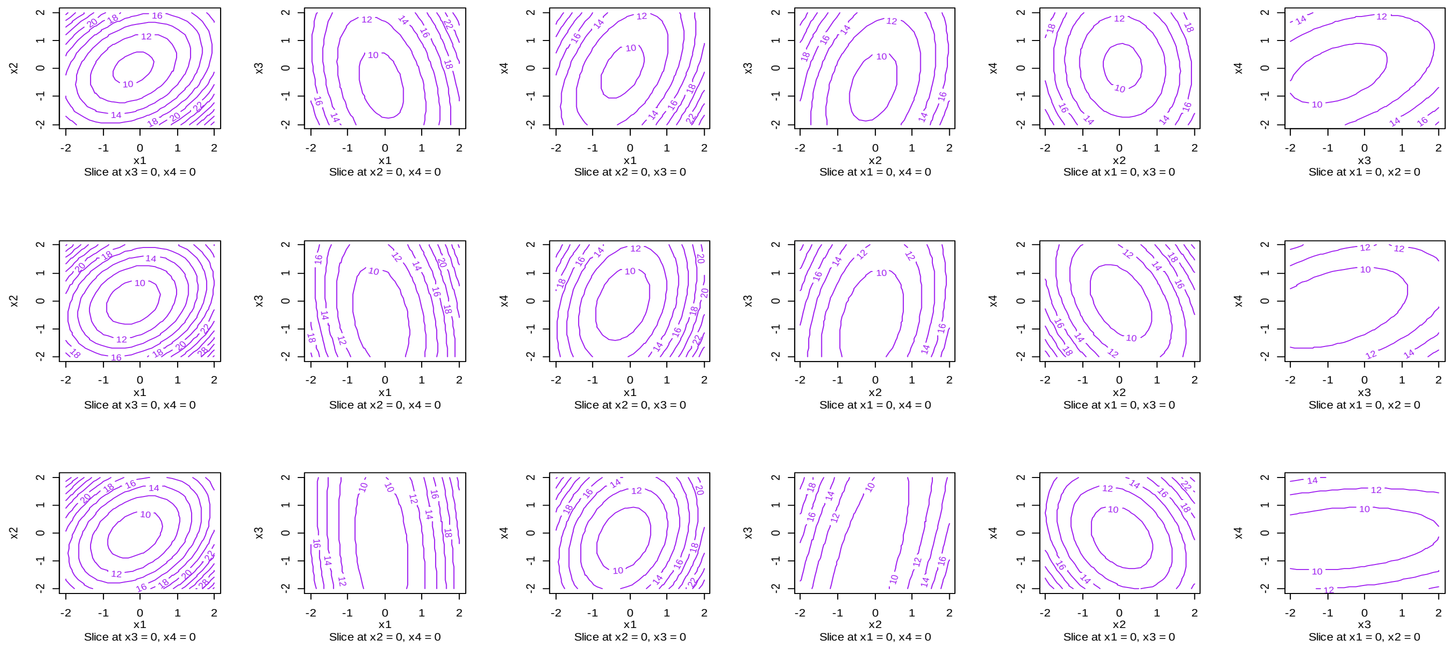


Figure 4: Contour Plot for the Minimum Response Model

**Analysis of the Saddle Response Model**

The characteristics of the fitted saddle response models are summarized in tables 8 and 9.

The tables show that all the three models were significant with p values of 0.0000, 0.0000 and 0.0035 for the three replicates

Table 8: The Fitted Saddle Response Model

| | Replicate I | | | | Replicate II | | | | Replicate III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Coeff. | Std. Error | t value | p value | Coeff. | Std. Error | t value | p value | Coeff. | Std. Error | t value | p value |
| Intercept | 10.167 | 0.468 | 21.703 | 0.000 | 10.000 | 0.585 | 17.108 | 0.000 | 10.833 | 0.862 | 12.564 | 0.000 |
| $x_1$ | 0.875 | 0.234 | 3.736 | 0.002 | 0.542 | 0.292 | 1.853 | 0.084 | 0.625 | 0.431 | 1.450 | 0.168 |
| $x_2$ | 0.625 | 0.234 | 2.668 | 0.018 | 0.625 | 0.292 | 2.139 | 0.049 | 0.542 | 0.431 | 1.256 | 0.228 |
| $x_3$ | 0.208 | 0.234 | 0.890 | 0.388 | 0.375 | 0.292 | 1.283 | 0.219 | 0.458 | 0.431 | 1.063 | 0.305 |
| $x_4$ | -0.042 | 0.234 | -0.178 | 0.861 | -0.042 | 0.292 | -0.143 | 0.889 | 0.042 | 0.431 | 0.097 | 0.924 |
| $x_1^2$ | -1.823 | 0.219 | -8.320 | 0.000 | -1.760 | 0.273 | -6.439 | 0.000 | -1.760 | 0.403 | -4.365 | 0.001 |
| $x_1x_2$ | 1.688 | 0.287 | 5.883 | 0.000 | 1.813 | 0.358 | 5.064 | 0.000 | 1.563 | 0.528 | 2.959 | 0.010 |
| $x_1x_3$ | 1.438 | 0.287 | 5.011 | 0.000 | 1.438 | 0.358 | 4.016 | 0.001 | 0.938 | 0.528 | 1.776 | 0.096 |
| $x_1x_4$ | 0.813 | 0.287 | 2.832 | 0.013 | 0.813 | 0.358 | 2.270 | 0.038 | 0.813 | 0.528 | 1.539 | 0.145 |
| $x_2^2$ | 1.677 | 0.219 | 7.655 | 0.000 | 1.740 | 0.273 | 6.363 | 0.000 | 1.490 | 0.403 | 3.694 | 0.002 |
| $x_2x_3$ | -0.688 | 0.287 | -2.397 | 0.030 | -0.438 | 0.358 | -1.222 | 0.240 | -0.438 | 0.528 | -0.829 | 0.420 |
| $x_2x_4$ | 0.188 | 0.287 | 0.654 | 0.523 | 0.188 | 0.358 | 0.524 | 0.608 | 0.188 | 0.528 | 0.355 | 0.727 |
| $x_3^2$ | 0.802 | 0.219 | 3.661 | 0.002 | 0.740 | 0.273 | 2.705 | 0.016 | 0.490 | 0.403 | 1.214 | 0.244 |
| $x_3x_4$ | 0.438 | 0.287 | 1.525 | 0.148 | 0.063 | 0.358 | 0.175 | 0.864 | 0.313 | 0.528 | 0.592 | 0.563 |
| $x_4^2$ | -0.698 | 0.219 | -3.185 | 0.006 | -0.510 | 0.273 | -1.867 | 0.082 | -0.510 | 0.403 | -1.266 | 0.225 |
| Multiple $R^2$ | 0.9482 | | | | 0.9174 | | | | 0.8030 | | | |
| Adjusted $R^2$ | 0.8999 | | | | 0.8403 | | | | 0.6230 | | | |
| F statistic | 19.620 | | | | 11.900 | | | | 4.424 | | | |
| P value | 0.0000 | | | | 0.0000 | | | | 0.0035 | | | |

Table 9: Analysis of Variance Table for the Saddle Response Model

| | Source | Sum of Squares | Degrees of Freedom | Mean Square | F value | P value |
|---|---|---|---|---|---|---|
| Replicate I | Model | 361.716 | 14 | 25.837 | 19.618 | 0.000 |
| | First Order | 28.833 | 4 | 7.208 | 5.475 | 0.006 |
| | Two way interaction | 100.375 | 6 | 16.729 | 12.706 | 0.000 |
| | Pure Quadratic | 232.508 | 4 | 58.127 | 44.147 | 0.000 |
| | Residuals | 19.750 | 15 | 1.317 | | |
| | Lack of fit | 12.917 | 10 | 1.292 | 0.945 | 0.563 |
| | Pure error | 6.833 | 5 | 1.367 | | |
| | Total | 381.466 | 29 | | | |
| Replicate II | Model | 341.416 | 14 | 24.387 | 11.896 | 0.000 |
| | First Order | 19.833 | 4 | 4.958 | 2.419 | 0.094 |
| | Two way interaction | 99.875 | 6 | 16.646 | 8.120 | 0.000 |
| | Pure Quadratic | 221.708 | 4 | 55.427 | 27.038 | 0.000 |
| | Residuals | 30.750 | 15 | 2.050 | | |
| | Lack of fit | 20.750 | 10 | 2.075 | 1.038 | 0.517 |
| | Pure error | 10.000 | 5 | 2.000 | | |
| | Total | 372.166 | 29 | | | |
| Replicate III | Model | 276.283 | 14 | 19.735 | 4.424 | 0.004 |
| | First Order | 21.500 | 4 | 5.375 | 1.205 | 0.349 |
| | Two way interaction | 68.875 | 6 | 11.479 | 2.573 | 0.064 |
| | Pure Quadratic | 185.908 | 4 | 46.477 | 10.418 | 0.000 |
| | Residuals | 66.917 | 15 | 4.461 | | |
| | Lack of fit | 50.083 | 10 | 5.008 | 1.488 | 0.346 |
| | Pure error | 16.833 | 5 | 3.367 | | |
| | Total | 343.200 | 29 | | | |

The stationary points, eigenvalues and canonical equivalent models are depicted in table 10:

Table 10: Stationary points and Eigenvalues

| | Stationary Points | | | | Eigenvalues | | | |
|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | | | |
| Replicate I | 0.020 | -0.234 | -0.216 | -0.117 | 1.909 | 1.035 | -0.708 | -2.277 |
| Replicate II | -0.062 | -0.170 | -0.238 | -0.136 | 1.976 | 0.943 | -0.464 | -2.246 |
| Replicate III | -0.068 | -0.198 | -0.433 | -0.182 | 1.690 | 0.633 | -0.503 | -2.112 |

Since for all the replicates the eigenvalues were of mixed sign, the response surface was a saddle one.
The canonical equivalent forms of the fitted models were respectively for the three replicates;

$$\hat{y} = 10.083 + 1.909w_1^2 + 1.035w_2^2 - 0.708w_3^2 - 2.277w_4^2 \quad (31)$$

$$\hat{y} = 9.888 + 1.976w_1^2 + 0.943w_2^2 - 0.464w_3^2 - 2.246w_4^2 \quad (32)$$

$$\hat{y} = 10.655 + 1.690w_1^2 + 0.633w_2^2 - 0.503w_3^2 - 2.112w_4^2 \quad (33)$$

The nature of these responses can be seen in the response surface plots shown in figure 5 and figure 6.

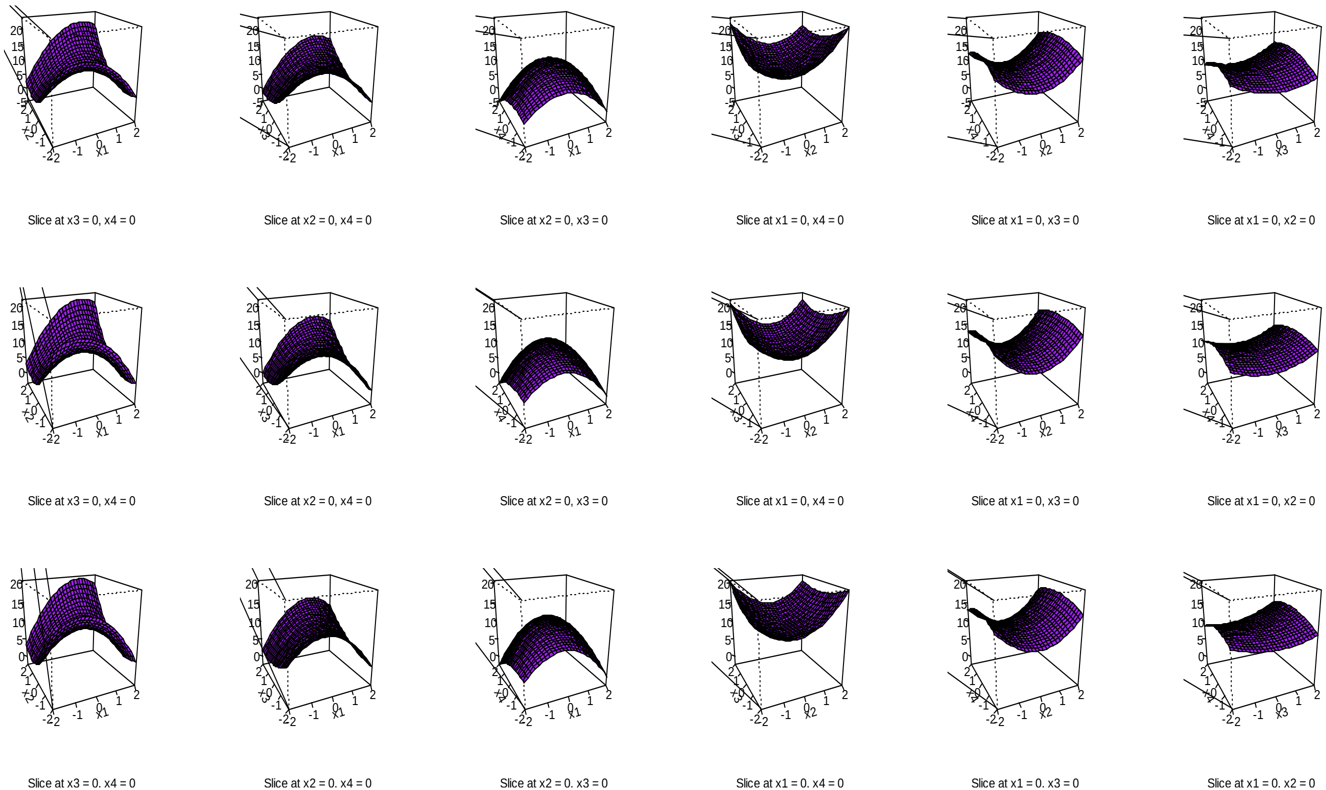


Figure 5: Response Surface Plot for the Saddle Response Model

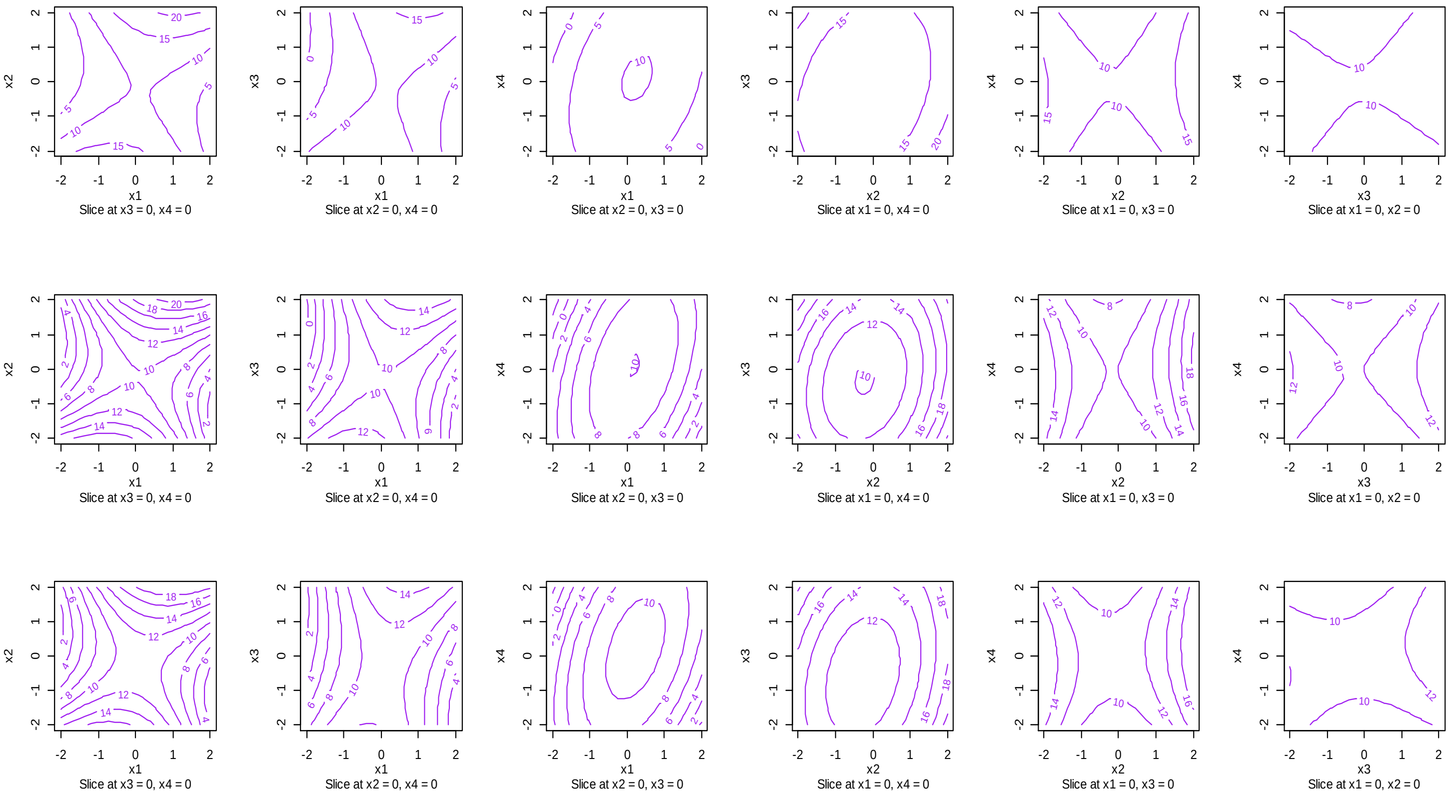


Figure 6: Contour Plot for the Saddle Response Model

**Application to the Germination of *Melia volkensii* Experiment**

**Description of the Germination of *Melia volkensii* Experiment**

A four factor rotatable central composite design was used in this experiment. The factors under investigation were temperature, soil PH, concentration of potassium nitrate ($KNO_3$) and length of time the seeds were soaked in $KNO_3$. The experiment was performed by soaking 20 seeds of Melia in a solution of $KNO_3$ for a specified period of time. They seeds were then placed in a petri-dish containing soil of a particular pH. They were then placed in germination chambers of a defined temperature. The outcome was the number of seeds that germinated in a particular petri-dish. The objective was to find the temperature, soil pH, concentration of $KNO_3$ and pre-treatment time that maximize the germination of Melia seeds.

The results of the experiment are presented in table 11.

Table 11: Germination of Melia Experiment Data

| Raw Values | | | | Coded Values | | | | Response |
|---|---|---|---|---|---|---|---|---|
| Temp. ($^oC$) | Soil PH | Concentration | Pre-treatment Time (Hours) | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $y$ |
| 20 | 5 | 0.2 | 6 | -1 | -1 | -1 | -1 | 4 |
| 30 | 5 | 0.2 | 6 | 1 | -1 | -1 | -1 | 3 |
| 20 | 9 | 0.2 | 6 | -1 | 1 | -1 | -1 | 5 |
| 30 | 9 | 0.2 | 6 | 1 | 1 | -1 | -1 | 1 |
| 20 | 5 | 0.4 | 6 | -1 | -1 | 1 | -1 | 3 |
| 30 | 5 | 0.4 | 6 | 1 | -1 | 1 | -1 | 6 |
| 20 | 9 | 0.4 | 6 | -1 | 1 | 1 | -1 | 6 |
| 30 | 9 | 0.4 | 6 | 1 | 1 | 1 | -1 | 5 |
| 20 | 5 | 0.2 | 10 | -1 | -1 | -1 | 1 | 1 |
| 30 | 5 | 0.2 | 10 | 1 | -1 | -1 | 1 | 6 |
| 20 | 9 | 0.2 | 10 | -1 | 1 | -1 | 1 | 3 |
| 30 | 9 | 0.2 | 10 | 1 | 1 | -1 | 1 | 4 |
| 20 | 5 | 0.4 | 10 | -1 | -1 | 1 | 1 | 0 |
| 30 | 5 | 0.4 | 10 | 1 | -1 | 1 | 1 | 5 |
| 20 | 9 | 0.4 | 10 | -1 | 1 | 1 | 1 | 4 |
| 30 | 9 | 0.4 | 10 | 1 | 1 | 1 | 1 | 10 |
| 15 | 7 | 0.3 | 8 | -2 | 0 | 0 | 0 | 0 |
| 35 | 7 | 0.3 | 8 | 2 | 0 | 0 | 0 | 3 |
| 25 | 3 | 0.3 | 8 | 0 | -2 | 0 | 0 | 1 |
| 25 | 11 | 0.3 | 8 | 0 | 2 | 0 | 0 | 4 |
| 25 | 7 | 0.1 | 8 | 0 | 0 | -2 | 0 | 7 |
| 25 | 7 | 0.5 | 8 | 0 | 0 | 2 | 0 | 6 |
| 25 | 7 | 0.3 | 4 | 0 | 0 | 0 | -2 | 7 |
| 25 | 7 | 0.3 | 12 | 0 | 0 | 0 | 2 | 8 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 5 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 8 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 12 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 10 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 7 |
| 25 | 7 | 0.3 | 8 | 0 | 0 | 0 | 0 | 11 |

**Analysis of the Germination of *Melia volkensii* Experiment**

The characteristics of the fitted models are summarized in tables 12 and 13.

Table 12: The Fitted Germination of *Melia volkensii* Model

| | Coefficient | Standard Error | t value | p value |
|---|---|---|---|---|
| (Intercept) | 8.833 | 0.732 | 12.075 | 0.000 |
| Temperature | 0.833 | 0.366 | 2.278 | 0.038 |
| Soil pH | 0.667 | 0.366 | 1.823 | 0.088 |
| Concentration | 0.417 | 0.366 | 1.139 | 0.273 |
| Time | 0.083 | 0.366 | 0.228 | 0.823 |
| Temperature$^2$ | -1.896 | 0.342 | -5.541 | 0.000 |
| Temperature: Soil pH | -0.625 | 0.448 | -1.395 | 0.183 |
| Temperature: Concentration | 0.750 | 0.448 | 1.674 | 0.115 |
| Temperature: Time | 1.250 | 0.448 | 2.790 | 0.014 |
| Soil pH$^2$ | -1.646 | 0.342 | -4.810 | 0.000 |
| Soil pH :Concentration | 0.750 | 0.448 | 1.674 | 0.115 |
| Soil pH : Time | 0.500 | 0.448 | 1.116 | 0.282 |
| Concentration$^2$ | -0.646 | 0.342 | -1.888 | 0.079 |
| Concentration: Time | -0.125 | 0.448 | -0.279 | 0.784 |
| Time$^2$ | -0.396 | 0.342 | -1.157 | 0.265 |
| Multiple $R^2$ | 0.8317 | | | |
| Adjusted $R^2$ | 0.6746 | | | |
| F statistic | 5.2940 | | | |
| P value | 0.0014 | | | |

Table 13: Analysis of Variance Table for the Germination of *Melia volkensii* Model

| Source | Sum of Squares | Degrees of Freedom | Mean Square | F value | P value |
|---|---|---|---|---|---|
| Model | 238.000 | 14 | 17.000 | 5.294 | 0.001 |
| First Order | 31.667 | 4 | 7.917 | 2.465 | 0.090 |
| Two way interaction | 53.500 | 6 | 8.917 | 2.777 | 0.051 |
| Pure Quadratic | 152.833 | 4 | 38.208 | 11.899 | 0.000 |
| Residuals | 48.167 | 15 | 3.211 | | |
| Lack of fit | 13.333 | 10 | 1.333 | 0.191 | 0.987 |
| Pure error | 34.833 | 5 | 6.967 | | |
| Total | 286.167 | 29 | | | |

The p value of 0.001 indicated that the model was significant. Further, the adjusted $\boldsymbol{R^2}$ value showed that 67.5% of the variability in the response was attributable to the model. The stationary point, eigenvalues and the canonical equivalent model are shown in table 14:

Table 14: Stationary point and Eigenvalues for the Germination of *Melia volkensii* Model

| Stationary Points | | | | Eigenvalues | | | |
|---|---|---|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | | | | |
| 0.869 | 0.507 | 0.962 | 1.646 | -0.134 | -0.511 | -1.463 | -2.476 |

Since all the eigenvalues were negative, the response surface was a maximum one.

The canonical equivalent form of the fitted *Melia volkensii* model was:

$$\hat{y} = 9.633 - 0.134w_1^2 - 0.511w_2^2 - 1.463w_3^2 - 2.476w_4^2 \quad (34)$$

The stationary point for the model was 0.869, 0.507, 0.962, 1.646. In terms of the natural variable this was 29.35, 8.01, 0.40, 11.29. Thus the optimal temperature was 29.35$^{o}$C, the optimal soil PH was 8.01, the optimal concentration of $KNO_3$ was 0.40% and the optimal pre-treatment time is 11.29 hours.
The nature of the response is displayed in figure 7 and figure 8.

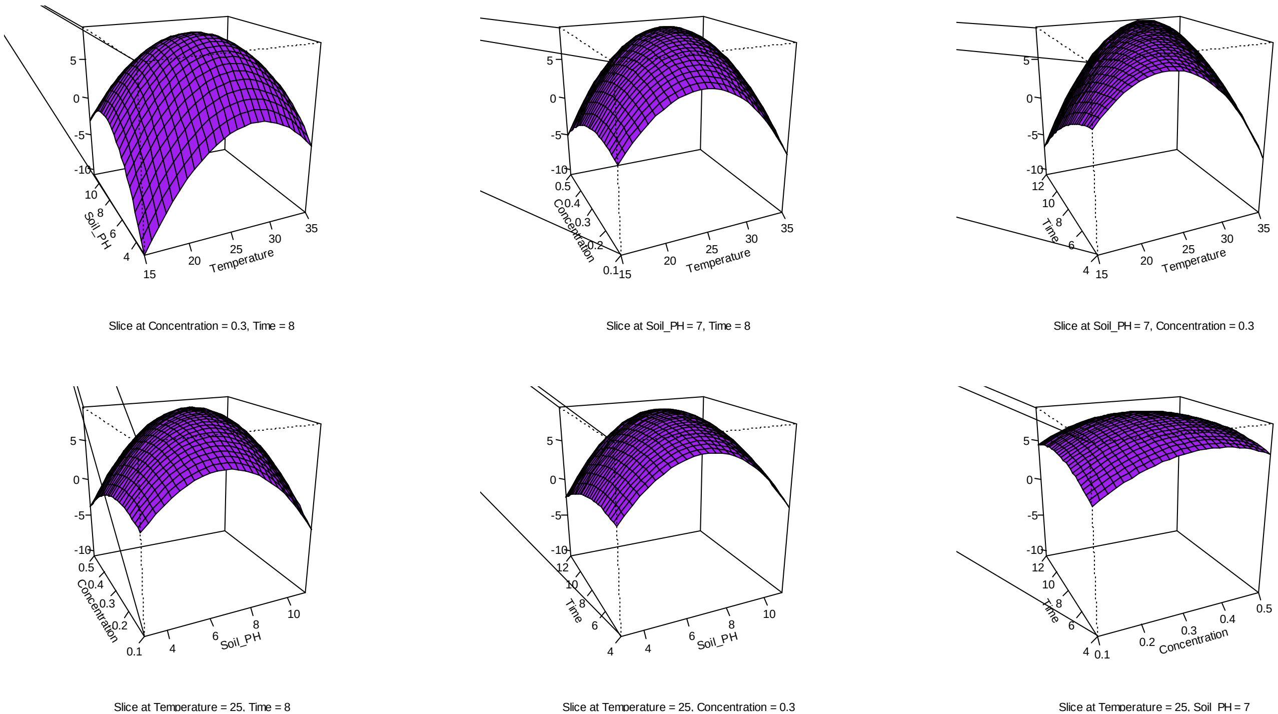


Figure 7: Response Surface Plot for the Germination of *Melia volkensii* Model

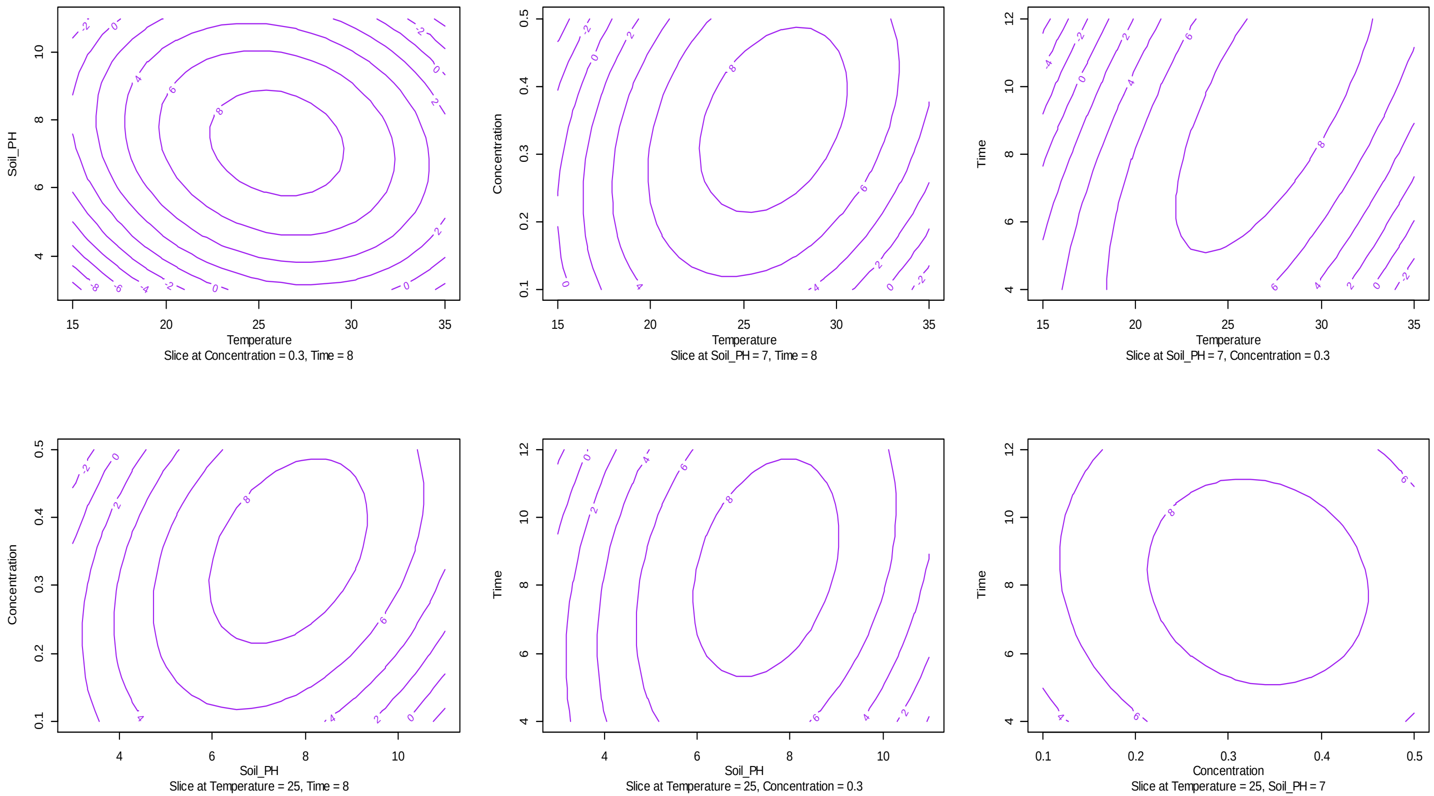


Figure 8: Contour Plot for the Germination of *Melia volkensii* Model

Suppose the investigator was interested in finding where to run the experiment to obtain a response that was close to 9 as possible. This could be obtained from the canonical equivalent model (34).

Letting $\hat{y} = 9$ gives

$$0.134w_1^2 + 0.511w_2^2 + 1.463w_3^2 + 2.476w_4^2 = 0.633 \qquad (35)$$

This region can is presented as a contour plot in figure 8

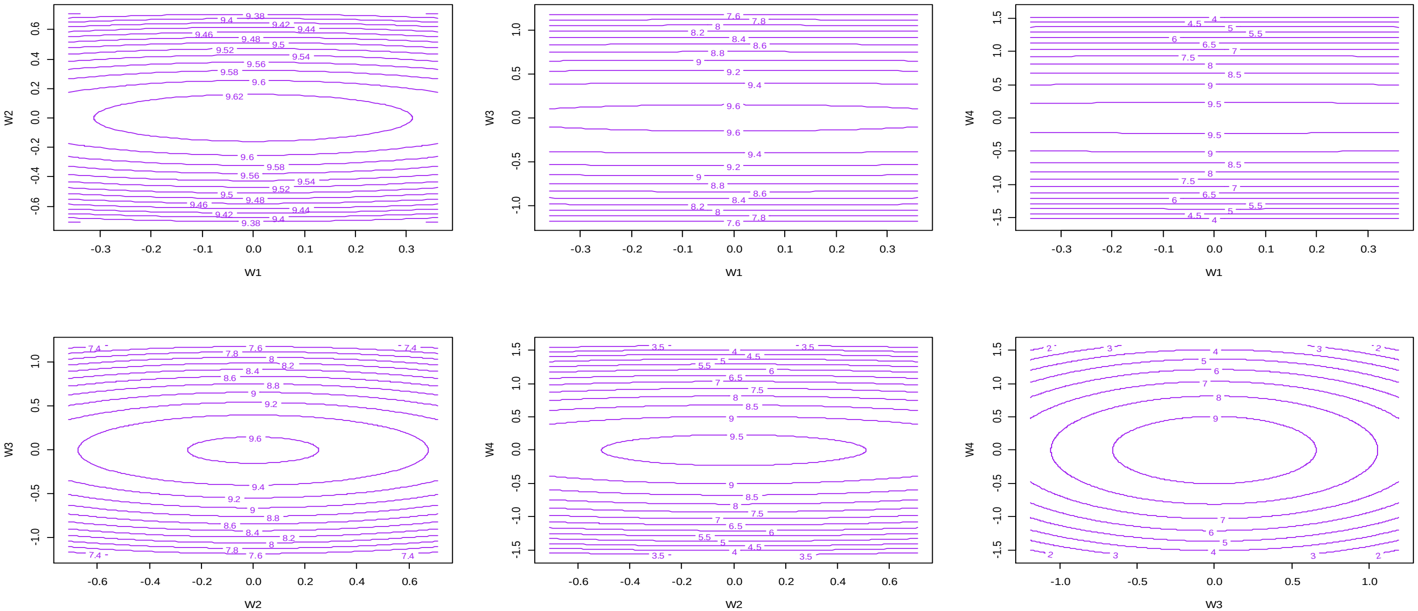


Figure 9: Contour Plot for Expected Response of 9

From (21)

$$\begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \\ x_3 - x_{3s} \\ x_4 - x_{4s} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

(36)

In this case

$$\begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{21} & m_{31} & m_{41} \\ m_{12} & m_{22} & m_{32} & m_{42} \\ m_{13} & m_{23} & m_{33} & m_{43} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}^{-1}, \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \\ x_{4s} \end{bmatrix} = \begin{bmatrix} 0.869 \\ 0.507 \\ 0.962 \\ 1.646 \end{bmatrix}$$ Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.869 \\ 0.507 \\ 0.962 \\ 1.646 \end{bmatrix} + \begin{bmatrix} 0.345 & 0.137 & 0.244 & 0.896 \\ -0.061 & -0.220 & -0.923 & 0.309 \\ 0.538 & -0.826 & 0.123 & -0.114 \\ -0.766 & -0.501 & 0.270 & 0.299 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

(37)

$$x_1 = 0.869 + 0.345w_1 + 0.137w_2 + 0.244w_3 + 0.896\, w_4$$

(38)

$$x_2 = 0.507 - 0.061w_1 - 0.220w_2 - 0.923w_3 + 0.309w_4$$

(39

$$x_3 = 0.962 + 0.538w_1 - 0.826w_2 + 0.123w_3 - 0.114w_4$$

(40)

$$x_4 = 1.646 - 0.766w_1 - 0.501w_2 + 0.270w_3 + 0.299w_4$$

(41)

The table 15 gives values of $w_1$, $w_2$, $w_3$, $w_4, x_1, x_2$ , $x_3$and $x_4$ for which $0.134w_1^2 + 0.511w_2^2 + 1.463w_3^2 + 2.476w_4^2 = 0.633$. When the experiment was run at the given values of the temperature, soil PH, concentration of $KNO_3$ and pre-treatment time the expected response was close to 9.

Table 15: Operating Conditions for Expected Response of 9

| $w_1$ | $w_2$ | $w_3$ | $w_4$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | Temp. | Soil PH | Conc. | Time | $\hat{y}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -0.2 | -0.2 | -0.5 | -0.5 | 0.203 | 0.870 | 1.015 | 1.615 | 26.01 | 8.74 | 0.40 | 11.23 | 8.2 |
| 0.2 | -0.2 | -0.5 | -0.5 | 0.341 | 0.846 | 1.230 | 1.309 | 26.70 | 8.69 | 0.42 | 10.62 | 8.6 |
| -0.2 | 0.2 | -0.5 | -0.5 | 0.257 | 0.782 | 0.685 | 1.415 | 26.29 | 8.56 | 0.37 | 10.83 | 8.5 |
| 0.2 | 0.2 | -0.5 | -0.5 | 0.395 | 0.758 | 0.900 | 1.108 | 26.98 | 8.52 | 0.39 | 10.22 | 8.8 |
| -0.2 | -0.2 | 0.5 | -0.5 | 0.447 | -0.053 | 1.138 | 1.885 | 27.23 | 6.89 | 0.41 | 11.77 | 7.8 |
| 0.2 | -0.2 | 0.5 | -0.5 | 0.585 | -0.077 | 1.353 | 1.579 | 27.92 | 6.85 | 0.44 | 11.16 | 8.0 |
| -0.2 | 0.2 | 0.5 | -0.5 | 0.501 | -0.141 | 0.808 | 1.685 | 27.51 | 6.72 | 0.38 | 11.37 | 8.1 |
| 0.2 | 0.2 | 0.5 | -0.5 | 0.639 | -0.165 | 1.023 | 1.378 | 28.20 | 6.67 | 0.40 | 10.76 | 8.3 |
| -0.2 | -0.2 | -0.5 | 0.5 | 1.099 | 1.179 | 0.901 | 1.914 | 30.49 | 9.36 | 0.39 | 11.83 | 8.3 |
| 0.2 | -0.2 | -0.5 | 0.5 | 1.237 | 1.155 | 1.116 | 1.608 | 31.18 | 9.31 | 0.41 | 11.22 | 8.1 |
| -0.2 | 0.2 | -0.5 | 0.5 | 1.153 | 1.091 | 0.571 | 1.714 | 30.77 | 9.18 | 0.36 | 11.43 | 8.0 |
| 0.2 | 0.2 | -0.5 | 0.5 | 1.291 | 1.067 | 0.786 | 1.407 | 31.46 | 9.13 | 0.38 | 10.81 | 7.8 |
| -0.2 | -0.2 | 0.5 | 0.5 | 1.343 | 0.256 | 1.024 | 2.184 | 31.71 | 7.51 | 0.40 | 12.37 | 8.8 |
| 0.2 | -0.2 | 0.5 | 0.5 | 1.481 | 0.232 | 1.239 | 1.878 | 32.40 | 7.46 | 0.42 | 11.76 | 8.5 |
| -0.2 | 0.2 | 0.5 | 0.5 | 1.397 | 0.168 | 0.694 | 1.984 | 31.99 | 7.34 | 0.37 | 11.97 | 8.6 |
| 0.2 | 0.2 | 0.5 | 0.5 | 1.535 | 0.144 | 0.909 | 1.677 | 32.68 | 7.29 | 0.39 | 11.35 | 8.2 |

## SUMMARY AND CONCLUSIONS

A comprehensive analysis of the four factor second order model was undertaken as reported in this paper. Cases for the three possible responses surfaces namely maximum response, minimum response and saddle response were considered. Additionally a practical experiment scenario was presented. The paper is therefore handy for researchers faced with the challenge of obtaining optimal values of four predictor variables when the response variable exhibits curvilinear behaviour. The work here can be extended with some appropriate modifications to any number of independent variables.

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