ASYMPTOTIC LINEAR ESTIMATION OF THE QUANTILE FUNCTION OF THE DOUBLE-EXPONENTIAL DISTRIBUTION BASED ON SELECTED ORDER STATISTICS

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Abstract. Asymptotic linear estimation of the quantile function for the location-scale family of distributions is considered and results relating to the double-exponential distribution discussed. A formula for the selection of sample quantiles for this distribution is derived. The ABLUE of the quantile function of the double exponential distribution and its variance are computed using two methods; Cheng(1975) and Ogawa (1998). These are analyzed to compare their behavior for various conditions.

Keywords. ABLUE, location-scale family, quantile function, double-exponential distribution.

1. Introduction

The location-scale family of distributions encompasses one of the largest family of distributions in probability theory. The double exponential distribution is one such member. A lot of work has been done on this class of distributions.

Ogawa (1951) obtained the asymptotically best linear unbiased estimator(ABLUE) of the scale parameter when the location parameter is known, then proposed a t-test based on a few selected order statistics to test the scale parameter of the exponential distribution Ogawa (1974).

Dixon (1957, 1960) proposed the simplified linear estimators for the mean and standard deviation of the normal population in terms of the sample quasi-midrange and quasi-range, respectively.

Epstein (1960) gave the BLUE of the quantile function for the one parameter exponential distribution in complete samples while Sarhan and Greenberg (1962) considered the one parameter case.

Hassanein (1968, 1972) considered the estimation of the quantile function for the Gumbel distribution for the large samples and Mann and Feitig (1977) for moderate samples.

Simplified estimation of parameters for a double exponential distribution was explored by Raghunandanan and Srinivasan(1971) while Cheng(1975) suggested a unified approach of obtaining optimal quantiles for ABLUEs.

Kubat and Epstein (1980) considered the estimation of the quantile function based on two or three order statistics for the normal and Gumbel distributions. Ali et al (1981) followed the approach of Kubat and Epstein (1980) for the exponential and double-exponential distributions based on two selected order statistics

Saleh and Rohatgi(1988) estimated the parameters of location scale family of distributions in complete and type II censored samples.

Ogawa (1998) gave a procedure of obtaining optimal spacing of the selected sample quantiles for the joint estimation of the location and scale parameters of a symmetric distribution.

Whereas a lot of work has been done for cases where up to three sample quantiles are involved, substantial work is yet to be done on the general case consisting of any number of selected quantiles. This paper will address this for the case of the double exponential distribution.

Weke (2005) obtained the ABLUE of the quantile function for the location scale exponential and logistic distributions based on any number of selected sample quantiles. This paper focuses on the ABLUE of the quantile function of the double exponential distribution.

2. The Location-Scale Family of Distributions

Suppose that a real-valued random variable Z has a continuous distribution with density g and distribution function G. Let a and b be constants with b>0.Then the distribution function of X= a + bZ is given by:

$$F(x)=G\left[\frac{(x-a)}{b}\right] \qquad (2.1)$$

The density function of X is given by:

$$f(x)=\frac{1}{b}\,g\left[\frac{(x-a)}{b}\right] \qquad (2.2)$$

In this case X is said to belong to a two-parameter location-scale family, where a is the location parameter and b is the scale parameter.

In the special case when b=1, we get the one parameter location family. Similarly in the special case that a=0, we get the one parameter scale family. The Standard form of the location – scale family of distributions is one in which the location parameter is zero while the scale parameter is one.

2.1 The Double Exponential distribution.

A random variable X has a double exponential (or Laplace) distribution if its probability density function (pdf) is:

$$f(x;a,b)=\frac{1}{2b}\exp(-|\frac{x-a}{b}|) \qquad -\infty<x<\infty \tag{2.3}$$

Where $|y|$ denotes the absolute value of y.

The cumulative distribution function is given by

$$F(x)=\begin{cases}\frac{1}{2}\exp(\frac{x-a}{b}) & if\ x<a\\ 1-\exp(-\frac{x-a}{b}) & if\ x\geq a\end{cases} \tag{2.4}$$

3. The Asymptotic Best Linear Unbiased Estimator (ABLUE) of Quantile Function in the Location-Scale Family of Distributions.

According to Ogawa (1951), for a given spacing $\{p_1, p_2 \ldots p_k\}$, the ABLUE of the quantile function, Q(ε) can be obtained through the generalized least-squares principle as

$$\tilde{Q}(\varepsilon)=\frac{1}{\Delta}\{(K_2X+Q_o(\varepsilon)K_1Y)-K_3(Q_o(\varepsilon)X)\} \tag{3.1}$$

where $\Delta=K_1K_2-K_3^2$ (3.2)

$$X=\sum_{i=1}^{k+1}\left\{\frac{d_o(p_i)-d_o(p_{i-1})}{p_i-p_{i-1}}-\frac{d_o(p_{i+1})-d_o(p_i)}{p_{i+1}-p_i}\right\}d_o(p_i)X_{(n_i)} \tag{3.3}$$

$$Y=\sum_{i=1}^{k+1}\left\{\frac{d_o(p_i)d_o(p_i)-d_o(p_{i-1})d_o(p_i)}{p_i-p_{i-1}}-\frac{Q_o(p_{i+1})d_o(p_{i+1})-Q_o(p_i)d_o(p_i)}{p_{i+1}-p_i}\right\}d_o(p_i)X_{(n_i)}$$

(3.4)

$$K_1=\sum_{i=1}^{k+1}\frac{\{d_o(p_i)-d_o(p_{i-1})\}^2}{p_i-p_{i-1}} \tag{3.5a}$$

$$K_2 = \sum_{i=1}^{k+1} \frac{\{Q_o(p_i)d_o(p_i) - Q_o(p_{i-1})d_o(p_i)\}^2}{p_i - p_{i-1}} \quad (3.5b)$$

$$K_3 = \sum_{i=1}^{k+1} \frac{\{d_o(p_i) - d_o(p_{i-1})\}\{Q_o(p_i)d_o(p_i) - Q_o(p_{i-1})d_o(p_{i-1})\}}{p_i - p_{i-1}} \quad (3.5c)$$

With $p_o=0$, $p_{k+1} = 1$, $d_o(p_o) = d_o(p_{k+1}) = 0$ and $n_i= [np_i]+1$, $i = 1,2,\ldots, k$ and $[y]$ denotes the greatest integer contained in y.

The variance of $\tilde{Q}(\varepsilon)$ is given by

$$\mathrm{Var}(\tilde{Q}(\varepsilon)) = \frac{\sigma^2}{n\Delta}\left\{K_2 + Q_o^2(\varepsilon)K_1 - 2Q_o(\varepsilon)K_3\right\} \quad (3.6)$$

If the pdf, f_o is symmetric about zero and if symmetric sample quantiles are selected i.e. $p_i+p_{k-i+1} =1$, $i=1, 2,\ldots, k$ then $K_3=0$ and the ABLUE of $Q(\varepsilon)$ is given by

$$\tilde{Q}(\varepsilon) = \frac{X}{K_1} + \frac{Q_o(\varepsilon)Y}{K_2} \quad (3.7)$$

With variance

$$\mathrm{Var}(\tilde{Q}(\varepsilon) = \frac{\sigma^2}{n}\left\{\frac{1}{K_1} + \frac{Q_o^2(\varepsilon)}{K_2}\right\} \quad (3.8)$$

When k=2 then the equations (3.3) through to (3.5c) become simpler as shown below:

From (3.3)

$$X=\left[\frac{p_2 d_o(p_1) - p_1 do(p_2)}{p_1(1-p_1)}\right] d_o(p_1)x_{(n_1)} + \left[\frac{(1-p_1)d_o(p_2) - (1-p_2)d_o(p_1)}{(p_2-p_1)(1-p_2)}\right] d_o(p_2)x_{(n_2)} \quad (3.9)$$

From (3.4)

$$Y = \left[\frac{d_o^2(p_{1)}}{p_1} - \frac{Q_o(p_2)d_o(p_2) - Q_o(p_1)d_o(p_1)}{p_2 - p_1}\right] d_o(p_1)x_{(n_1)} +$$
$$\left[\frac{d_o^2(p_2) - d_o(p_1)d_o(p_2)}{p_2 - p_1} + \frac{Q_o(p_2)d_o(p_2)}{1-p_2}\right] d_o(p_2)x_{(n_2)} \quad (3.10)$$

From (3.5a)

$$K_1 = \frac{p_2 d_o^2(p_1) + p_1 d_o^2(p_2) - 2p_1 d_o(p_1) d_o(p_2)}{p_1(p_2 - p_1)} \tag{3.11a}$$

From (3.5b)

$$K_2 = \frac{(Q_o(p_1)d_o(p_1))^2}{p_1} + \frac{(Q_o(p_2)d_o(p_2) - Q_o(p_1)d_o(p_2))^2}{p_2 - p_1} \tag{3.11b}$$

From (3.5c)

$$K_3 = \frac{d_o^2(p_1)Q_o(p_1)}{p_1} + \frac{[d_o(p_2) - d_o(p_1)][Q_o(p_2)d_o(p_2) - Q_o(p_1)d_o(p_1)]}{p_2 - p_1} \tag{3.11c}$$

When k≥3 X, Y, K_1, K_2 and K_3 become complex but can be computed by use of an appropriate algorithm.

3.1 The Asymptotic Best Linear Unbiased Estimator (ABLUE) of the Quantile Function for the Double Exponential Distribution.

Let $x_{(n_1)} < x_{(n_2)} < x_{(n_3)} < \ldots < x_{(n_k)}$ be k arbitrary but fixed order statistics with ranks n_i= $[np_i] + 1$, i = 1, 2,…,k in a sample of size n from the double exponential distribution whose distribution function is:

$$F(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{\mu - x}{\sigma}\right) & if\ x < \mu \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\mu}{\sigma}\right) & if\ x \geq \mu \end{cases} \quad where\ \sigma > 0 \tag{3.12}$$

This can be compactly expressed as:

$$F(x) = 0.5[1+\text{sgn}(x-\mu)(1-\exp(-|x-\mu|/\sigma))] \tag{3.13}$$

where sgn is the sign function and is given by

$$\operatorname{sgn}(a) = \begin{cases} + & if \quad a \geq 0 \\ - & if \quad a < 0 \end{cases}$$

The inverse cumulative function is given by:

$$F^{-1}(p)=\mu-\sigma \operatorname{sgn}(p-0.5)\ln(1-2|p-0.5|) \tag{3.14}$$

for $0<p<1$

For the location scale model, the cumulative distribution function (cdf) has the form

$$F(x) = F_o\left(\frac{x-\mu}{\sigma}\right) \tag{3.15}$$

where F_o is a known distribution and μ and σ are the unknown location and scale parameters respectively.

The probability density function (pdf) is then given by

$$f(x) = \frac{1}{\sigma} f_o\left(\frac{x-\mu}{\sigma}\right) \tag{3.16}$$

where $F_o'=f_o$

The quantile function is defined as

$$Q(\varepsilon) = F^{-1}(\varepsilon) \tag{3.17}$$

Using the location property

$$Q(\varepsilon) = \mu + \sigma Q_o(\varepsilon) \tag{3.18}$$

where $Q_o(\varepsilon) = F_o^{-1}(\varepsilon)$

From (3.14) and (3.17) we get

$$Q(\varepsilon) = \mu-\sigma \operatorname{sgn}(\varepsilon-0.5)\ln(1-2|\varepsilon-0.5|) \tag{3.19}$$

Equating the coefficients of (3.18) and (3.19) leads to

$$Q_o(\varepsilon) = -\operatorname{sgn}(\varepsilon-0.5)\ln(1-2|\varepsilon-0.5|) \tag{3.20}$$

The probability density function for the double exponential distribution is given by

$$f(x)=\frac{1}{2\sigma}\exp\left[-\operatorname{sgn}(x-\mu)\left(\frac{x-u}{\sigma}\right)\right] \tag{3.21}$$

Comparing (3.16) and (3.21) we have

$$f_o\left(\frac{x-\mu}{\sigma}\right)=\frac{1}{2}\exp\left[-\text{sgn}(x-\mu)\left(\frac{x-\mu}{\sigma}\right)\right] \quad (3.22)$$

Equation (3.22) can be used to evaluate the density, d_o at a particular value, p_i of the quantile function. i.e. $d_o(p_i)=f_o(Q_o(p_i))$, with $Q_o(p_i)$ computed using (3.20)

In order to obtain the ABLUE of the quantile function Q (ε) for k selected order statistics, it is required that (3.6) be minimized subject to the restriction $0<p_1<p_2< \ldots <p_k<1$.

According to Cheng (1975) the unified approach for finding the optimal spacing is as follows:

The spacing $\{\lambda_i^c\}$ is the unique solution of the system of equations

$$M(\lambda_{i-1}, \lambda_i, \lambda_{i+1})=0 \quad i = 1, 2, \ldots ,k \quad (3.23)$$

if it can be written as

$$H_1(y_{i-1}) = H_2(y_i) \quad i = 1, 2, \ldots, k \quad (3.24)$$

by the substitution

$$y_i = \frac{h(\lambda_i)}{h(\lambda_{i+1})} \quad y_0=0,\ 0<y_i<1,\ i = 0,1,2, \ldots, k \quad (3.25)$$

or

$$y_i = \frac{h^*(\lambda_{k-i+1)}}{h^*(\lambda_{k-i})} \quad y_0=0,\ 0<y_i<1,\ i = 0,1,2 \ldots k \quad (3.26)$$

where h(λ) and h*(λ) are both monotonic functions of λ.

$H_1(y)$ and $H_2(y)$ are such that for any fixed $y'\in[0,1)$ there exists a unique $y^*\in(y',1)$ such that

$$H_2(y^*) = H_1(y') \quad (3.27)$$

if

$$H_2(y)>H_1(y) \text{ for all } y \in (0,1) \quad (3.28)$$

and

$$H_1(1) = H_2(1) = \text{a constant} \quad (3.29)$$

For the double exponential distribution, substitution in the unified approach results gives:

$h^*(\lambda) = 1-\lambda$ (3.30)

$$H_1(y) = 1+\frac{y\ln y}{2(1-y)} \quad (3.31)$$

$$H_2(y) = \frac{-\ln y}{2(1-y)} \quad (3.32)$$

From (3.24) we get

$H_1(y_{i-1}) = H_2(y_i)$ (3.33)

Combining (3.30) , (3.31), (3.32) and (3.33) we have:

$$1-\frac{\left(\frac{1-\lambda_{k-i+1}}{1-\lambda_{k-i}}\right)\ln\left(\frac{1-\lambda_{k-i+1}}{1-\lambda_{k-i}}\right)}{2\left(1-\frac{1-\lambda_{k-i+1}}{1-\lambda_{k-i}}\right)} = -\frac{\ln\left(\frac{1-\lambda_{k-i}}{1-\lambda_{k-i-1}}\right)}{2\left(1-\frac{1-\lambda_{k-i}}{1-\lambda_{k-i-1}}\right)} \quad (3.34)$$

With a given value of λ_1 , (3.34) can then be used to iteratively compute the optimal spacing $\{\lambda_i\}$ i = 0 , 1 , 2 ,..., k+1 with $\lambda_0 = 0$ and $\lambda_{k+1} = 1$.

An alternative method of finding the optimal spacing is to select symmetric sample quantiles using the formula

$\lambda_i+\lambda_{k-i+1} = 1.$ (3.35)

(See Ogawa, 1998)

In this way (3.34) will be used to generate the first half of the required points in the spacing then the other half will be computed using (3.35).

4. Simulation Results.

Given a random variable U drawn from the uniform distribution in the interval $[-\frac{1}{2},\frac{1}{2}]$, the variable

$X = \mu - \sigma\, \mathrm{sgn}(U)\ln(1-2|U|)$ (4.1)

has a double exponential distribution with parameters μ and σ. This follows from the inverse cumulative distribution function given in (3.14).

In the standard case where $\mu=0$ and $\sigma=1$ we get

$$X = -\operatorname{sgn}(U)\ln(1-2|U|) \qquad (4.2)$$

Thus the random variable U was generated from the uniform distribution in the interval $[-\frac{1}{2},\frac{1}{2}]$ and using (4.2) the variable X was generated. This procedure was used to generate a sample of size 100 from the standard double exponential distribution

The sample values were then arranged in ascending order of magnitude to obtain the requisite order statistics. Two methods of selecting the optimal spacing were used. The first method is due to Cheng (1975). Using this method requires finding solutions to (3.34). In order to obtain the spacing $p_1, p_2, \ldots, p_k$ some initial conditions are required . Here a value of p_1 must be set so that the other values of p can be computed.

The second method of selecting the optimal spacing is to have symmetric quantiles i.e. $p_i+p_{k-i+1}=1$. In this case the first half of the required spacing was generated using Chengs method described above whilst the remaining half was determined by the relationship $p_i+p_{k-i+1}=1$.

Once the optimal spacing has been obtained it is then possible to generate the sample quantiles $x_{(n_1)} < x_{(n_2)} < x_{(n_3)} < \ldots < x_{(n_k)}$ since $n_i = [np_i+1]$. Having obtained the sample quantiles, the ABLUE of the quantile function and the variance of the ABLUE were computed. Equations (3.1) and (3.6) provide the formulas used to compute the ABLUE of the quantile function and the variance of the ABLUE respectively.

The results for the ABLUE of the quantile function are presented in Table 4.1 and Table 4.2 whereas those of the variance of the ABLUE are in Table 4.3 and 4.4 .

Table 4.1: The ABLUE of the quantile function using the unified approach method, Cheng(1975).

						ε				
p_1	**k**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
	10	523.1200	297.8250	166.0350	72.5290	0.0000	-72.5290	-166.0350	-297.8250	-523.1200
	15	37.5463	21.3760	11.9170	5.2057	0.0000	-5.2057	-11.9170	-21.3760	-37.5463
	20	20.7801	11.8306	6.5955	2.8811	0.0000	-2.8811	-6.5955	-11.8306	-20.7801
	25	14.3654	8.1786	4.5595	1.9917	0.0000	-1.9917	-4.5595	-8.1786	-14.3654
	30	10.7377	6.1132	3.4081	1.4888	0.0000	-1.4888	-3.4081	-6.1132	-10.7377
	35	8.3686	4.7645	2.6562	1.1603	0.0000	-1.1603	-2.6562	-4.7645	-8.3686
	40	6.7361	3.8350	2.1380	0.9339	0.0000	-0.9339	-2.1380	-3.8350	-6.7361
0.01	45	5.4967	3.1294	1.7446	0.7621	0.0000	-0.7621	-1.7446	-3.1294	-5.4967
	50	4.6968	2.6740	1.4907	0.6512	0.0000	-0.6512	-1.4907	-2.6740	-4.6968
	55	4.1751	2.3770	1.3252	0.5789	0.0000	-0.5789	-1.3252	-2.3770	-4.1751
	60	3.8680	2.2021	1.2277	0.5363	0.0000	-0.5363	-1.2277	-2.2021	-3.8680
	65	3.9357	2.2600	1.2798	0.5843	0.0449	-0.4946	-1.1901	-2.1703	-3.8459
	70	4.3294	2.4832	1.4032	0.6370	0.0426	-0.5518	-1.3180	-2.3980	-4.2443
	75	5.0875	2.9140	1.6425	0.7404	0.0407	-0.6590	-1.5611	-2.8325	-5.0060
	80	5.9545	3.4069	1.9166	0.8593	0.0391	-0.7810	-1.8384	-3.3286	-5.8762
	10	0.2350	0.1338	0.0746	0.0326	0.0000	-0.0326	-0.0746	-0.1338	-0.2350
	15	0.5492	0.3238	0.1920	0.0985	0.0259	-0.0466	-0.1402	-0.2720	-0.4974
	20	1.3337	0.7686	0.4381	0.2036	0.0217	-0.1602	-0.3947	-0.7252	-1.2902
	25	1.7082	0.9807	0.5552	0.2533	0.0191	-0.2151	-0.5170	-0.9425	-1.6700
	30	1.4449	0.8301	0.4704	0.2152	0.0173	-0.1806	-0.4358	-0.7954	-1.4103
0.05	35	0.7146	0.4138	0.2378	0.1130	0.0161	-0.0807	-0.2055	-0.3815	-0.6823
	40	-0.4864	-0.2703	-0.1439	-0.0542	0.0154	0.0850	0.1747	0.3011	0.5172
	45	-1.7775	-1.0055	-0.5539	-0.2335	0.0150	0.2636	0.5840	1.0356	1.8076
	50	-2.7160	-1.5399	-0.8519	-0.3638	0.0148	0.3935	0.8816	1.5696	2.7457
	55	-2.9908	-1.6964	-0.9392	-0.4019	0.0148	0.4315	0.9688	1.7260	3.0204
	60	-3.0029	-1.7033	-0.9430	-0.4036	0.0148	0.4332	0.9726	1.7328	3.0325
	10	0.5604	0.3348	0.2028	0.1092	0.0365	-0.0361	-0.1297	-0.2617	-0.4873
	15	0.4694	0.2799	0.1690	0.0903	0.0293	-0.0317	-0.1104	-0.2213	-0.4109
0.10	20	-0.6044	-0.3328	-0.1740	-0.0612	0.0262	0.1136	0.2264	0.3852	0.6568
	25	-1.6658	-0.9375	-0.5114	-0.2091	0.0253	0.2598	0.5621	0.9881	1.7165
	30	-1.7429	-0.9814	-0.5359	-0.2199	0.0253	0.2704	0.5865	1.0319	1.7934

0.15	10	0.1482	0.0915	0.0583	0.0348	0.0165	-0.0017	-0.0253	-0.0584	-0.1151
	15	-1.3153	-0.7426	-0.4076	-0.1699	0.0145	0.1988	0.4365	0.7715	1.3443
	20	-1.5531	-0.8781	-0.4832	-0.2030	0.0143	0.2316	0.5118	0.9067	1.5818
0.20	10	-0.9127	-0.5167	-0.2850	-0.1206	0.0069	0.1344	0.2988	0.5304	0.9265
	15	-1.4633	-0.8302	-0.4599	-0.1971	0.0067	0.2105	0.4732	0.8436	1.4766

Table 4.2: The ABLUE of the quantile function using the symmetric method. Ogawa(1998)

					ε					
p_1	**k**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.01	10	-0.9145	-0.7021	-0.5779	-0.4897	-0.4214	-0.3530	-0.2649	-0.1406	0.0717
	15	-1.1740	-0.8012	-0.5831	-0.4284	-0.3084	-0.1884	-0.0337	0.1844	0.5572
	20	-0.3569	-0.3181	-0.2953	-0.2792	-0.2667	-0.2542	-0.2380	-0.2153	-0.1764
	25	-0.2668	-0.2206	-0.1935	-0.1743	-0.1594	-0.1446	-0.1254	-0.0983	-0.0521
	30	0.1381	0.0206	-0.0481	-0.0969	-0.1347	-0.1725	-0.2213	-0.2900	-0.4075
	35	0.2684	0.1028	0.0059	-0.0628	-0.1161	-0.1694	-0.2382	-0.3351	-0.5007
	40	0.4968	0.2227	0.0623	-0.0515	-0.1397	-0.2280	-0.3418	-0.5022	-0.7763
	45	0.8370	0.4477	0.2199	0.0583	-0.0670	-0.1923	-0.3539	-0.5816	-0.9710
	50	0.9767	0.5079	0.2338	0.0392	-0.1117	-0.2626	-0.4571	-0.7313	-1.2000
	55	1.4116	0.8026	0.4463	0.1935	-0.0025	-0.1986	-0.4514	-0.8076	-1.4167
	60	1.5562	0.8708	0.4699	0.1855	-0.0352	-0.2558	-0.5403	-0.9412	-1.6266
	65	2.0020	1.1415	0.6381	0.2809	0.0039	-0.2732	-0.6303	-1.1337	-1.9943
	70	2.2296	1.2816	0.7270	0.3335	0.0283	-0.2769	-0.6703	-1.2249	-2.1730
	75	2.5916	1.4795	0.8289	0.3674	0.0094	-0.3487	-0.8102	-1.4608	-2.5729
	80	2.6529	1.4875	0.8058	0.3221	-0.0531	-0.4282	-0.9119	-1.5936	-2.7590
0.05	10	-0.5425	-0.3504	-0.2379	-0.1582	-0.0963	-0.0345	0.0453	0.1577	0.3499
	15	-0.0531	-0.0325	-0.0205	-0.0119	-0.0053	0.0013	0.0099	0.0219	0.0425
	20	0.1741	0.1080	0.0692	0.0418	0.0205	-0.0008	-0.0283	-0.0670	-0.1332
	25	0.4159	0.2435	0.1426	0.0711	0.0156	-0.0399	-0.1115	-0.2123	-0.3847
	30	0.7228	0.4196	0.2422	0.1164	0.0188	-0.0788	-0.2047	-0.3820	-0.6852
	35	0.8144	0.4733	0.2737	0.1322	0.0223	-0.0875	-0.2291	-0.4286	-0.7697
	40	1.1106	0.6371	0.3601	0.1636	0.0112	-0.1412	-0.3378	-0.6147	-1.0882
	45	1.1408	0.6722	0.3981	0.2036	0.0528	-0.0980	-0.2925	-0.5666	-1.0352
	50	1.2241	0.7215	0.4275	0.2189	0.0571	-0.1047	-0.3133	-0.6073	-1.1099
	55	1.2187	0.7117	0.4151	0.2046	0.0414	-0.1218	-0.3322	-0.6288	-1.1359
	60	1.1511	0.6731	0.3935	0.1951	0.0413	-0.1126	-0.3110	-0.5906	-1.0685
	65	1.2114	0.7056	0.4098	0.1999	0.0370	-0.1258	-0.3357	-0.6315	-1.1373

	70	1.1152	0.6513	0.3799	0.1873	0.0380	-0.1114	-0.3039	-0.5753	-1.0392
	75	1.2607	0.7373	0.4311	0.2139	0.0454	-0.1231	-0.3404	-0.6465	-1.1699
	80	1.1081	0.6497	0.3816	0.1914	0.0438	-0.1038	-0.2940	-0.5622	-1.0205
	10	-0.1305	-0.0622	-0.0222	0.0061	0.0281	0.0501	0.0784	0.1184	0.1866
	15	0.2144	0.1320	0.0837	0.0495	0.0230	-0.0036	-0.0378	-0.0860	-0.1685
	20	0.5249	0.3155	0.1930	0.1061	0.0387	-0.0287	-0.1156	-0.2381	-0.4474
	25	0.4748	0.2948	0.1894	0.1147	0.0567	-0.0013	-0.0760	-0.1813	-0.3614
	30	0.3731	0.2357	0.1553	0.0983	0.0540	0.0098	-0.0473	-0.1277	-0.2651
0.10	35	0.4071	0.2493	0.1569	0.0914	0.0406	-0.0102	-0.0757	-0.1681	-0.3259
	40	0.2565	0.1705	0.1202	0.0845	0.0568	0.0291	-0.0065	-0.0568	-0.1428
	45	0.3162	0.1891	0.1147	0.0620	0.0211	-0.0199	-0.0726	-0.1469	-0.2740
	50	0.1724	0.1119	0.0764	0.0513	0.0318	0.0123	-0.0128	-0.0482	-0.1088
	55	0.1903	0.1172	0.0745	0.0441	0.0206	-0.0029	-0.0333	-0.0760	-0.1491
	60	0.1885	0.1159	0.0733	0.0432	0.0198	-0.0036	-0.0338	-0.0763	-0.1490
	10	0.2913	0.1730	0.1038	0.0546	0.0165	-0.0215	-0.0707	-0.1399	-0.2582
	15	0.2249	0.1491	0.1048	0.0734	0.0490	0.0246	-0.0068	-0.0511	-0.1268
	20	0.0849	0.0778	0.0736	0.0706	0.0683	0.0660	0.0630	0.0588	0.0516
0.15	25	0.1430	0.1061	0.0845	0.0692	0.0573	0.0454	0.0301	0.0085	-0.0284
	30	-0.1512	-0.0634	-0.0120	0.0245	0.0528	0.0811	0.1176	0.1690	0.2568
	35	-0.1107	-0.0531	-0.0195	0.0044	0.0230	0.0415	0.0654	0.0990	0.1566
	40	-0.1144	-0.0566	-0.0228	0.0012	0.0199	0.0385	0.0625	0.0963	0.1541
	10	0.1619	0.1134	0.0850	0.0649	0.0493	0.0337	0.0135	-0.0148	-0.0633
	15	-0.0219	0.0071	0.0241	0.0362	0.0455	0.0549	0.0670	0.0840	0.1130
0.20	20	-0.2590	-0.1185	-0.0364	0.0219	0.0671	0.1124	0.1706	0.2528	0.3932
	25	-0.2653	-0.1443	-0.0735	-0.0233	0.0156	0.0545	0.1048	0.1755	0.2965
	30	-0.2712	-0.1512	-0.0810	-0.0312	0.0074	0.0461	0.0959	0.1661	0.2861

Table 4.3: The variance of the ABLUE of the quantile function using the unified approach method. Cheng(1975)

		ε								
p_1	**k**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.01	10	0.1049	1.0939	2.5152	3.9009	5.1914	6.6703	8.8550	12.4659	20.0794
	15	0.0839	0.0821	0.1296	0.1851	0.2404	0.3067	0.4081	0.5816	0.9611
	20	0.0750	0.0536	0.0655	0.0848	0.1061	0.1328	0.1754	0.2507	0.4213
	25	0.0693	0.0442	0.0460	0.0547	0.0657	0.0804	0.1049	0.1497	0.2547
	30	0.0654	0.0395	0.0371	0.0411	0.0475	0.0567	0.0728	0.1036	0.1779
	35	0.0627	0.0368	0.0321	0.0336	0.0374	0.0436	0.0551	0.0779	0.1351
	40	0.0609	0.0350	0.0291	0.0290	0.0312	0.0355	0.0441	0.0620	0.1082
	45	0.0596	0.0339	0.0270	0.0259	0.0271	0.0301	0.0367	0.0512	0.0901
	50	0.0587	0.0331	0.0257	0.0237	0.0242	0.0263	0.0315	0.0436	0.0772
	55	0.0582	0.0326	0.0247	0.0222	0.0221	0.0235	0.0276	0.0379	0.0675
	60	0.0580	0.0324	0.0240	0.0211	0.0205	0.0214	0.0247	0.0336	0.0601
	65	0.0546	0.0304	0.0226	0.0199	0.0194	0.0204	0.0237	0.0324	0.0581
	70	0.0501	0.0278	0.0208	0.0186	0.0184	0.0197	0.0232	0.0322	0.0578
	75	0.0467	0.0257	0.0194	0.0175	0.0176	0.0190	0.0228	0.0319	0.0575
	80	0.0439	0.0241	0.0182	0.0167	0.0169	0.0185	0.0224	0.0315	0.0569
0.05	10	0.0517	0.0311	0.0251	0.0235	0.0237	0.0254	0.0294	0.0389	0.0653
	15	0.0417	0.0244	0.0192	0.0177	0.0178	0.0191	0.0223	0.0299	0.0515
	20	0.0336	0.0192	0.0152	0.0144	0.0150	0.0165	0.0199	0.0276	0.0485
	25	0.0303	0.0170	0.0134	0.0126	0.0132	0.0146	0.0178	0.0249	0.0442
	30	0.0289	0.0161	0.0124	0.0116	0.0119	0.0131	0.0159	0.0224	0.0400
	35	0.0286	0.0158	0.0120	0.0110	0.0111	0.0121	0.0145	0.0203	0.0364
	40	0.0288	0.0159	0.0119	0.0107	0.0106	0.0114	0.0135	0.0187	0.0338
	45	0.0294	0.0162	0.0120	0.0106	0.0103	0.0109	0.0128	0.0176	0.0319
	50	0.0299	0.0165	0.0121	0.0105	0.0102	0.0107	0.0124	0.0170	0.0308
	55	0.0303	0.0167	0.0122	0.0106	0.0102	0.0106	0.0122	0.0167	0.0304
	60	0.0303	0.0167	0.0122	0.0106	0.0102	0.0106	0.0122	0.0167	0.0303
0.10	10	0.0336	0.0189	0.0148	0.0140	0.0145	0.0161	0.0196	0.0274	0.0485
	15	0.0293	0.0160	0.0123	0.0113	0.0117	0.0129	0.0157	0.0222	0.0402
	20	0.0295	0.0161	0.0119	0.0105	0.0104	0.0112	0.0133	0.0187	0.0341
	25	0.0307	0.0167	0.0121	0.0104	0.0101	0.0105	0.0123	0.0171	0.0314
	30	0.0310	0.0169	0.0122	0.0105	0.0101	0.0105	0.0122	0.0169	0.0310
0.15	10	0.0313	0.0168	0.0126	0.0116	0.0119	0.0132	0.0164	0.0235	0.0432
	15	0.0326	0.0174	0.0124	0.0107	0.0104	0.0110	0.0131	0.0186	0.0347

	20	0.0335	0.0179	0.0127	0.0108	0.0103	0.0108	0.0126	0.0178	0.0334
0.20	10	0.0355	0.0183	0.0129	0.0111	0.0109	0.0118	0.0144	0.0210	0.0403
	15	0.0374	0.0193	0.0133	0.0111	0.0106	0.0111	0.0133	0.0192	0.0371

Table 4.4: The variance of the ABLUE of the quantile function using the symmetric method. Ogawa(1998)

						ε				
p_1	**k**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0.01	10	0.2049	0.1424	0.1217	0.1141	0.1124	0.1141	0.1217	0.1424	0.2049
	15	0.1432	0.0958	0.0802	0.0744	0.0731	0.0744	0.0802	0.0958	0.1432
	20	0.1133	0.0734	0.0602	0.0554	0.0542	0.0554	0.0602	0.0734	0.1133
	25	0.0928	0.0594	0.0484	0.0443	0.0434	0.0443	0.0484	0.0594	0.0928
	30	0.0804	0.0505	0.0406	0.0370	0.0362	0.0370	0.0406	0.0505	0.0804
	35	0.0703	0.0438	0.0351	0.0319	0.0312	0.0319	0.0351	0.0438	0.0703
	40	0.0637	0.0391	0.0310	0.0281	0.0274	0.0281	0.0310	0.0391	0.0637
	45	0.0578	0.0353	0.0278	0.0251	0.0245	0.0251	0.0278	0.0353	0.0578
	50	0.0537	0.0324	0.0253	0.0228	0.0222	0.0228	0.0253	0.0324	0.0537
	55	0.0499	0.0299	0.0233	0.0209	0.0203	0.0209	0.0233	0.0299	0.0499
	60	0.0472	0.0279	0.0216	0.0193	0.0187	0.0193	0.0216	0.0279	0.0472
	65	0.0446	0.0262	0.0202	0.0180	0.0174	0.0180	0.0202	0.0262	0.0446
	70	0.0426	0.0248	0.0190	0.0168	0.0163	0.0168	0.0190	0.0248	0.0426
	75	0.0408	0.0236	0.0179	0.0158	0.0153	0.0158	0.0179	0.0236	0.0408
	80	0.0394	0.0226	0.0170	0.0150	0.0145	0.0150	0.0170	0.0226	0.0394
0.05	10	0.0542	0.0333	0.0264	0.0239	0.0233	0.0239	0.0264	0.0333	0.0542
	15	0.0403	0.0239	0.0185	0.0165	0.0160	0.0165	0.0185	0.0239	0.0403
	20	0.0342	0.0192	0.0142	0.0124	0.0120	0.0124	0.0142	0.0192	0.0342
	25	0.0324	0.0178	0.0129	0.0111	0.0107	0.0111	0.0129	0.0178	0.0324
	30	0.0310	0.0164	0.0116	0.0098	0.0094	0.0098	0.0116	0.0164	0.0310
	35	0.0286	0.0148	0.0103	0.0086	0.0082	0.0086	0.0103	0.0148	0.0286
	40	0.0269	0.0138	0.0095	0.0079	0.0075	0.0079	0.0095	0.0138	0.0269
	45	0.0247	0.0127	0.0087	0.0073	0.0069	0.0073	0.0087	0.0127	0.0247
	50	0.0233	0.0120	0.0082	0.0069	0.0065	0.0069	0.0082	0.0120	0.0233
	55	0.0216	0.0112	0.0078	0.0065	0.0062	0.0065	0.0078	0.0112	0.0216
	60	0.0206	0.0107	0.0074	0.0063	0.0060	0.0063	0.0074	0.0107	0.0206
	65	0.0195	0.0102	0.0072	0.0060	0.0058	0.0060	0.0072	0.0102	0.0195
	70	0.0189	0.0099	0.0070	0.0059	0.0056	0.0059	0.0070	0.0099	0.0189
	75	0.0182	0.0096	0.0068	0.0057	0.0055	0.0057	0.0068	0.0096	0.0182
	80	0.0179	0.0094	0.0067	0.0057	0.0054	0.0057	0.0067	0.0094	0.0179
0.10	10	0.0349	0.0189	0.0136	0.0116	0.0112	0.0116	0.0136	0.0189	0.0349
	15	0.0328	0.0170	0.0118	0.0099	0.0095	0.0099	0.0118	0.0170	0.0328
	20	0.0293	0.0147	0.0098	0.0081	0.0077	0.0081	0.0098	0.0147	0.0293
	25	0.0240	0.0122	0.0083	0.0069	0.0066	0.0069	0.0083	0.0122	0.0240

	30	0.0217	0.0111	0.0076	0.0063	0.0060	0.0063	0.0076	0.0111	0.0217
	35	0.0194	0.0101	0.0071	0.0059	0.0057	0.0059	0.0071	0.0101	0.0194
	40	0.0184	0.0097	0.0068	0.0057	0.0055	0.0057	0.0068	0.0097	0.0184
	45	0.0178	0.0094	0.0066	0.0056	0.0054	0.0056	0.0066	0.0094	0.0178
	50	0.0176	0.0093	0.0065	0.0055	0.0053	0.0055	0.0065	0.0093	0.0176
	55	0.0175	0.0092	0.0065	0.0055	0.0053	0.0055	0.0065	0.0092	0.0175
	60	0.0174	0.0092	0.0065	0.0055	0.0053	0.0055	0.0065	0.0092	0.0174
0.15	10	0.0426	0.0221	0.0153	0.0128	0.0122	0.0128	0.0153	0.0221	0.0426
	15	0.0287	0.0144	0.0096	0.0079	0.0075	0.0079	0.0096	0.0144	0.0287
	20	0.0239	0.0121	0.0082	0.0067	0.0064	0.0067	0.0082	0.0121	0.0239
	25	0.0199	0.0104	0.0073	0.0061	0.0059	0.0061	0.0073	0.0104	0.0199
	30	0.0189	0.0100	0.0070	0.0059	0.0057	0.0059	0.0070	0.0100	0.0189
	35	0.0185	0.0098	0.0069	0.0058	0.0056	0.0058	0.0069	0.0098	0.0185
	40	0.0183	0.0097	0.0069	0.0058	0.0056	0.0058	0.0069	0.0097	0.0183
0.20	10	0.0454	0.0211	0.0130	0.0101	0.0094	0.0101	0.0130	0.0211	0.0454
	15	0.0249	0.0126	0.0086	0.0071	0.0068	0.0071	0.0086	0.0126	0.0249
	20	0.0213	0.0111	0.0077	0.0064	0.0061	0.0064	0.0077	0.0111	0.0213
	25	0.0197	0.0104	0.0073	0.0062	0.0059	0.0062	0.0073	0.0104	0.0197
	30	0.0194	0.0103	0.0073	0.0062	0.0059	0.0062	0.0073	0.0103	0.0194

5. Discussion.

Table 4.1 and Table 4.2 show the computed results for the ABLUE of the quantile function for the double exponential distribution using two methods of selecting the sample quantiles whereas Table 4.3 and Table 4.4 provide the respective variances of these estimates. It is desired that for a particular estimate the variance be minimum. We will now discuss how to select sample quantiles to achieve minimum variance.

5.1. Effect of the Number of Sample Quantiles (k) on the Variance of the ABLUE of Quantile Function.

We investigate the number sample quantiles to be selected for the estimate to have minimum variance. Figure 5.1 through to Figure 5.6 below show how the variance varies with the k.

Figure 5.1. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.01(Cheng Method)

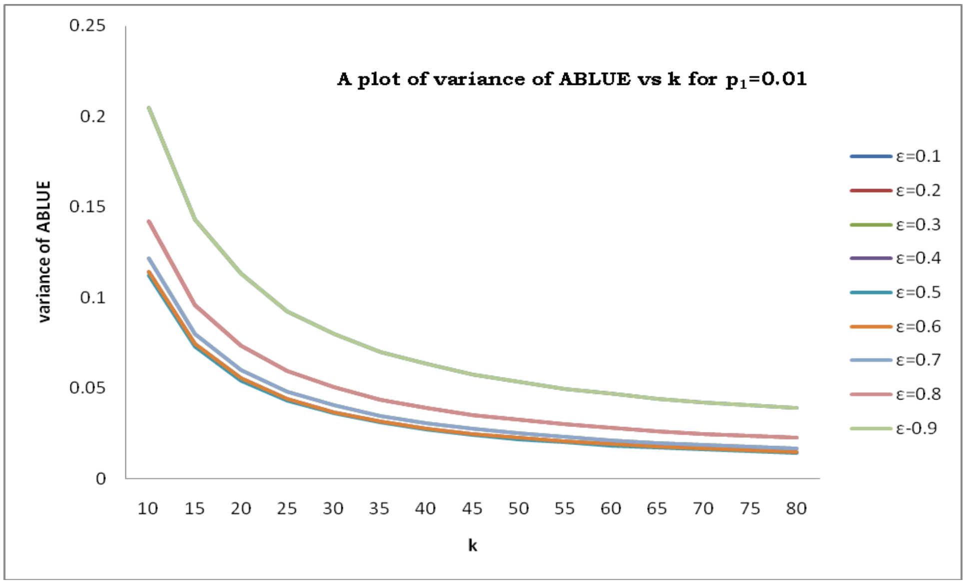


Figure 5.2. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.05 (Cheng Method)

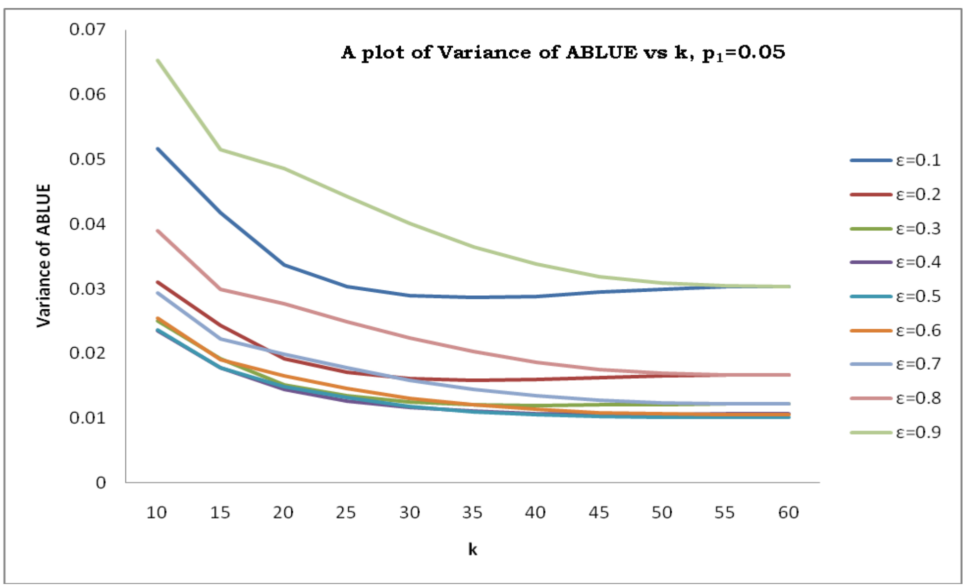


Figure 5.3. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.10 (Cheng Method)

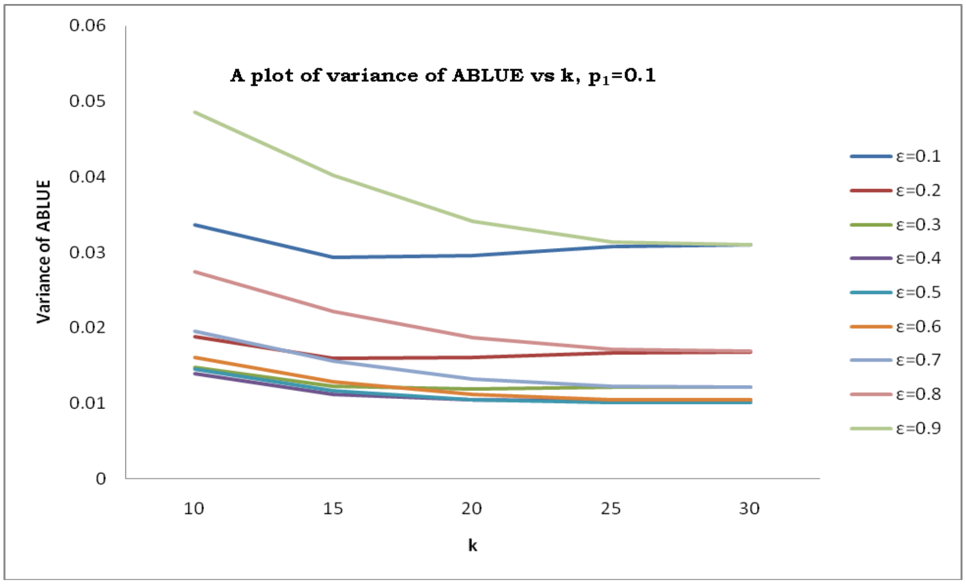


Figure 5.4. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.01(Ogawa Method)

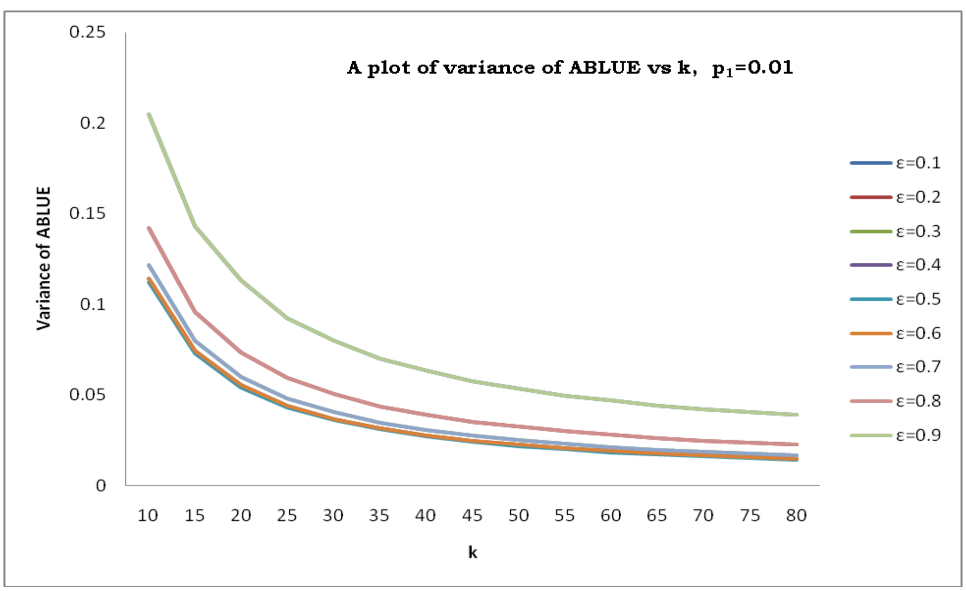


Figure 5.5. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.10 (Ogawa Method)

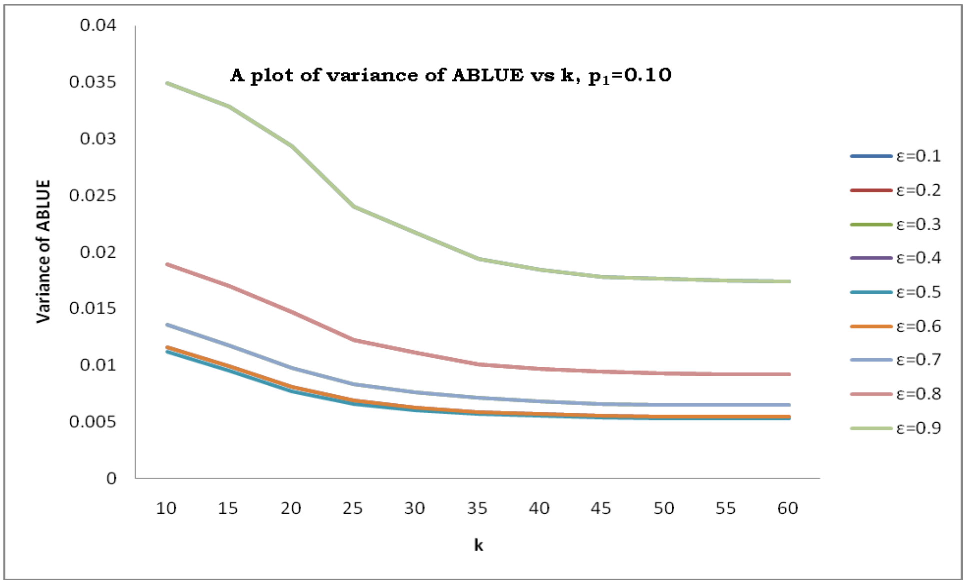
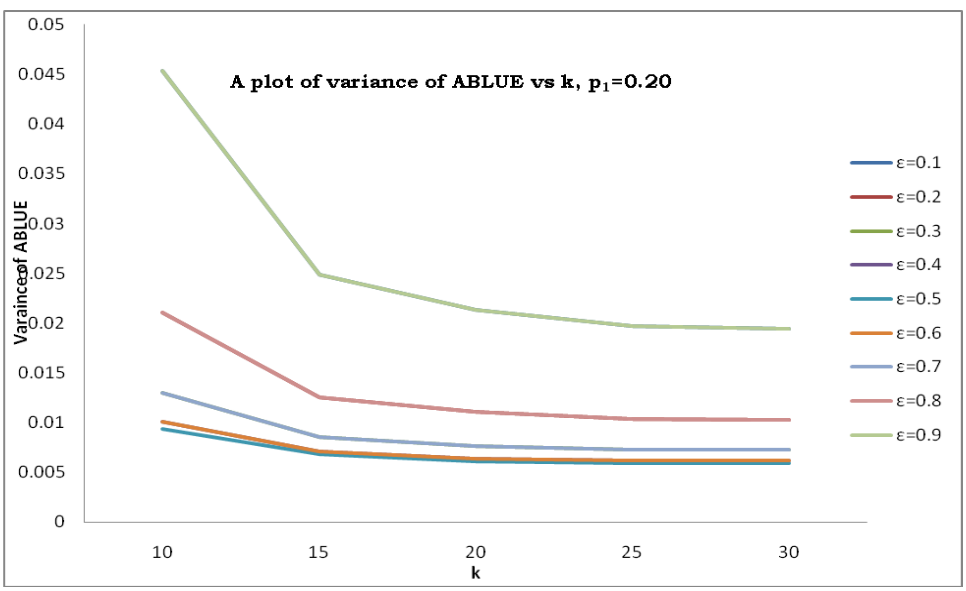


Figure 5.6. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.20(Ogawa Method)



From the six graphs above, it can be seen that the variance of the ABLUE of the quantile function is in most cases a decreasing function of k. In such cases the best value of k to be used is one beyond which the variance does not decrease any further. However in some cases the variance decreases then begins to rise. The best value of k for such cases is one in which the variance is a minimum.

5.2. Effect of p_1 on the Variance of the ABLUE of Quantile Function.

The chosen value of p_1 will determine whether the selected sample quantiles will be left censored, right censored or doubly censored. In this section we investigate the effect of such a selection on the variance of the ABLUE of the quantile function.

Figure 5.7 through to Figure 5.10 depicts how p_1 affects the variance of the ABLUE of the quantile function.

Figure 5.7. A graph of the variance of the ABLUE of the quantile function vs p_1 , k=10 (Cheng Method)

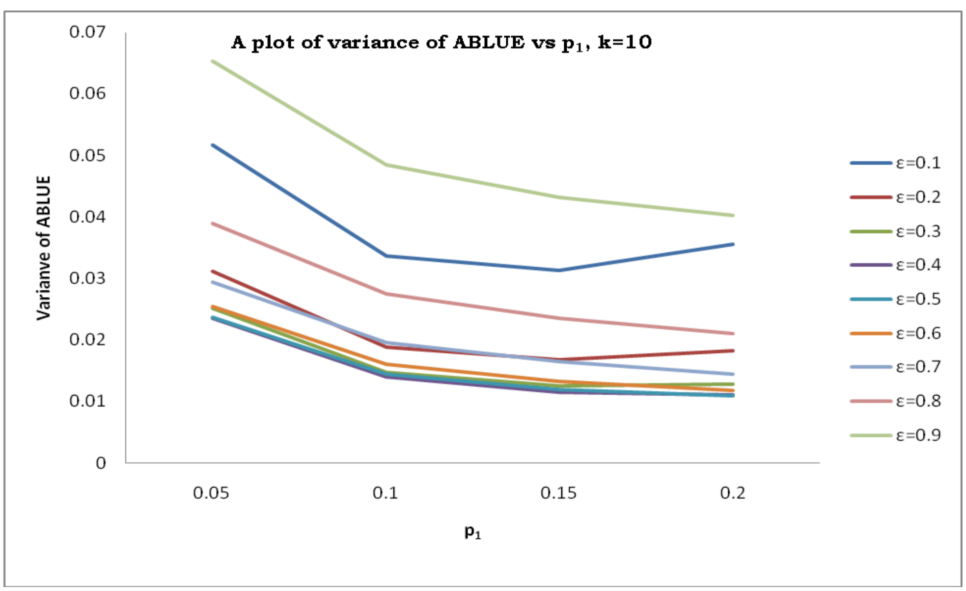


Figure 5.8. A graph of the variance of the ABLUE of the quantile function vs p_1 , k=30 (Cheng Method)

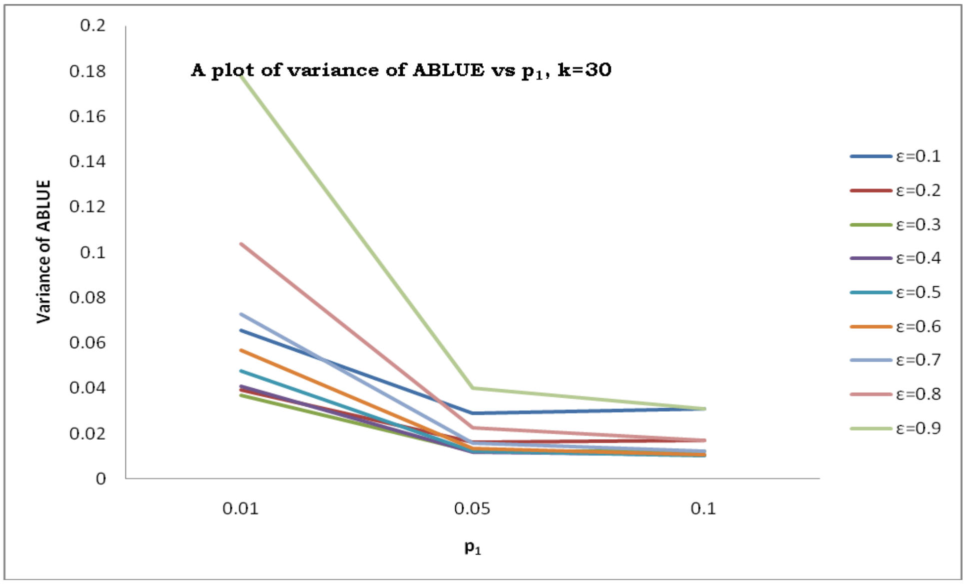


Figure 5.9. A graph of the variance of the ABLUE of the quantile function vs p_1 , k=10 (Ogawa Method)

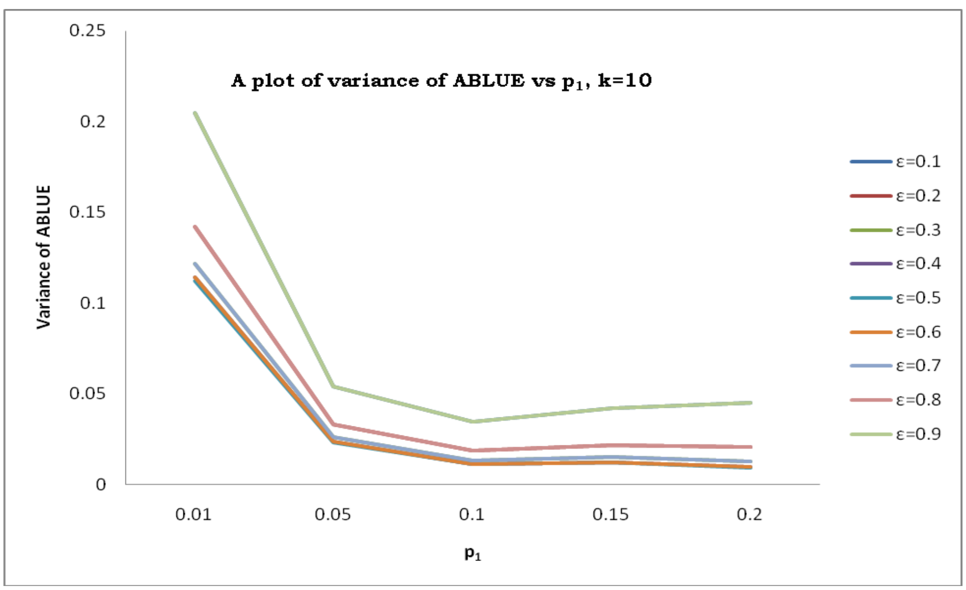


Figure 5.10. A graph of the variance of the ABLUE of the quantile function vs p_1, k=30 (Ogawa Method)

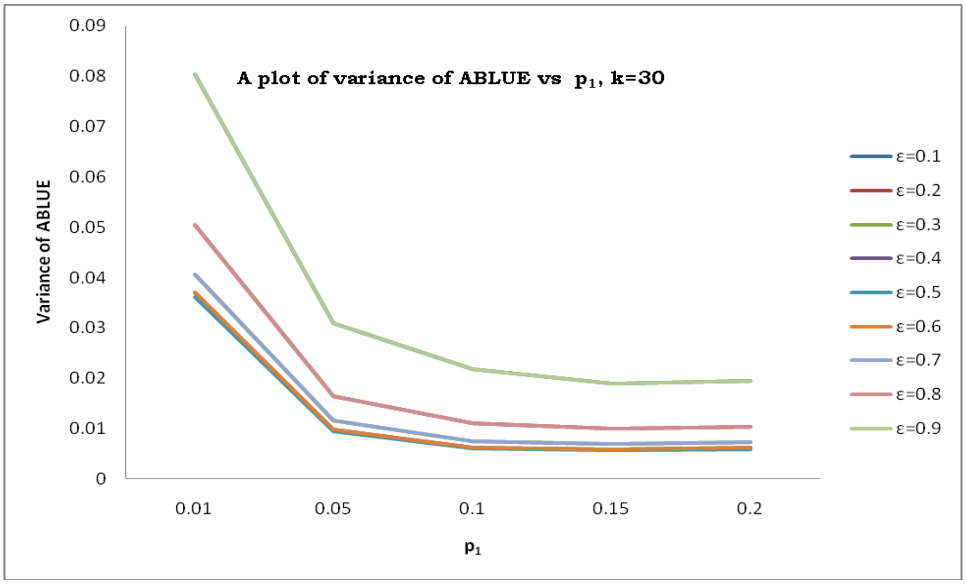


Figure 5.7, Figure 5.8, Figure 5.9 and Figure 5.10 indicate in general that the lower the value of p_1 the higher the variance. In most cases the variances stable at values of p_1 between 0.05 and 0.15

5.3. Effect of the Method of Selection of the Sample Quantiles on the Variance of the ABLUE of the Quantile Function.

We are now going to assess how the different methods of selecting the sample quantiles affect the variance of the ABLUE of the quantile function. This is shown in the following figures.

Figure 5.11. A graph of the variance of the ABLUE of the quantile function vs ε p_1=0.01, k=50

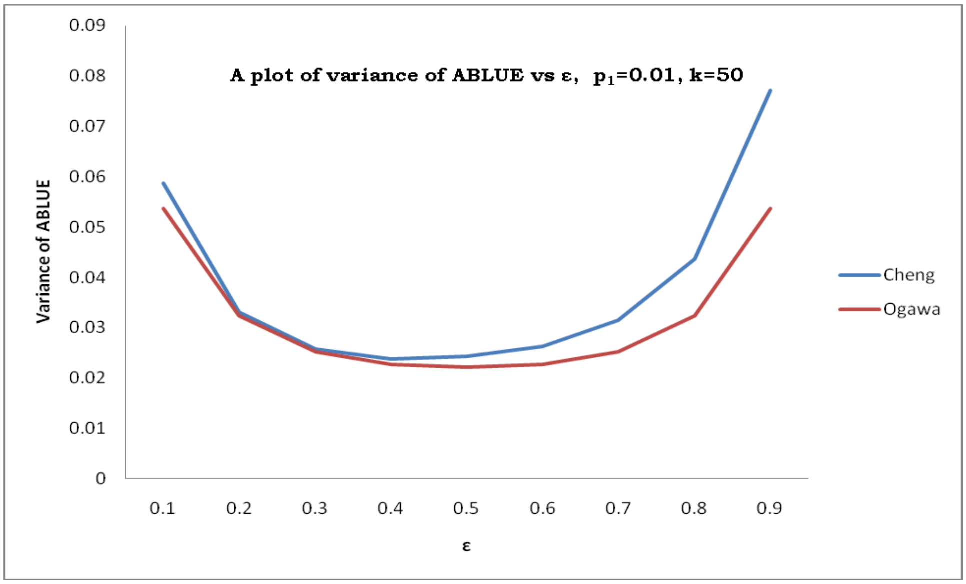


Figure 5.12. A graph of the variance of the ABLUE of the quantile function vs ε p_1=0.05, k=40

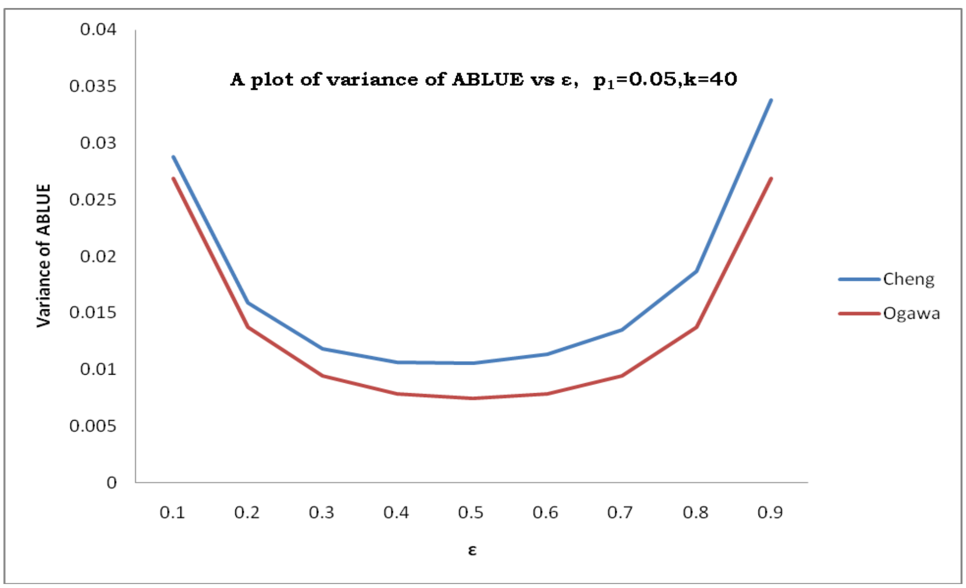


Figure 5.13. A graph of the variance of the ABLUE of the quantile function vs ε p_1=0.10, k=20

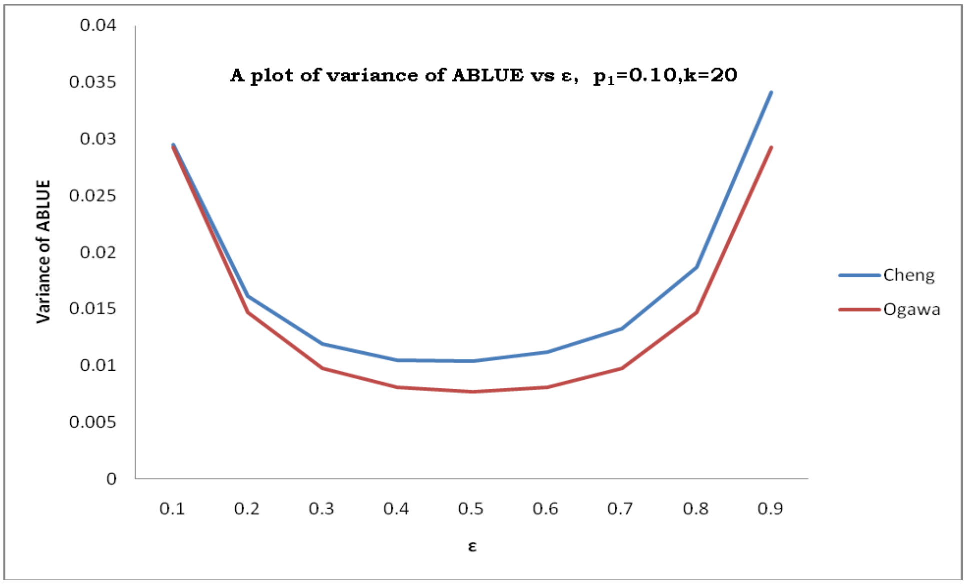


Figure 5.14. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.01, ε =0.1

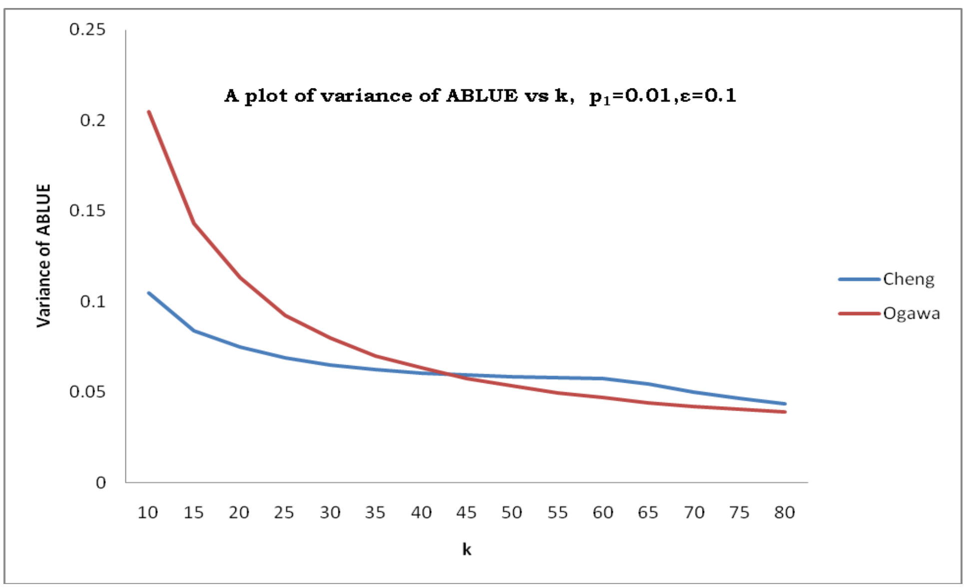


Figure 5.15. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.05, ε =0.5

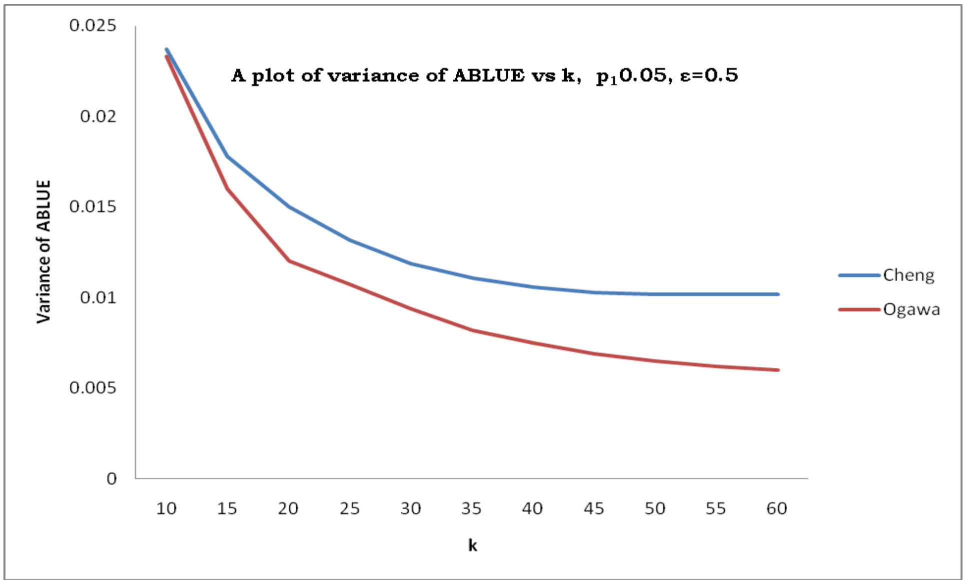


Figure 5.16. A graph of the variance of the ABLUE of the quantile function vs k, p_1=0.10, ε =0.9

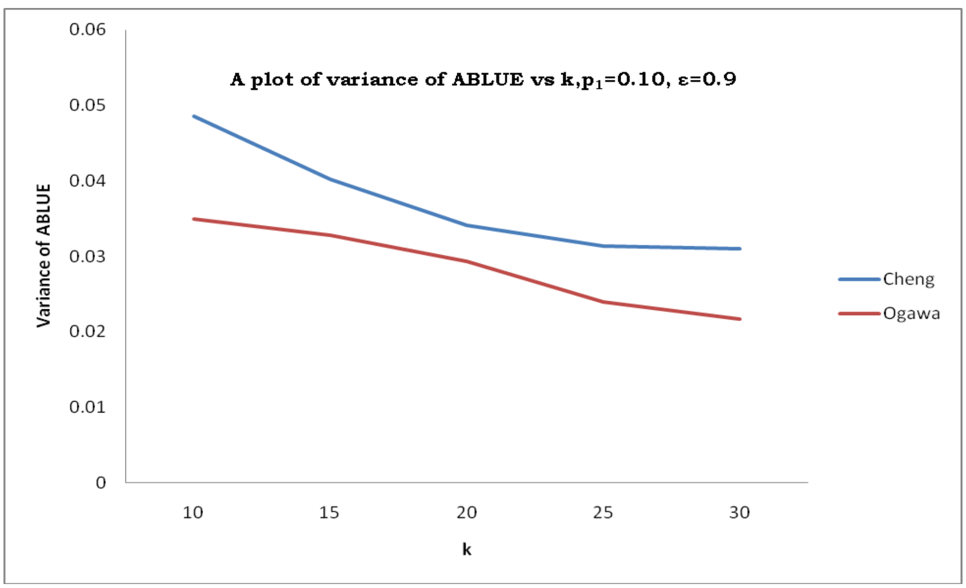


Figure 5.11 through to Figure 5.16 indicate that in most cases the method by (Ogawa, 1998) produces sample quantiles for which the variance of the quantile function has lower variances than those of (Cheng, 1975). However Figure 5.14 shows otherwise. Therefore to determine the best method to be used for selecting sample quantiles the number to be selected and the value of p_1 also need to be considered.

6. Summary and Conclusions

The main aim of this paper was to investigate the use of selected order statistics in evaluating the quantile function. When selecting these order statistics a number of questions need to be addressed. These include; The number of order statistics to be selected, whether the selected sample quantiles will be censored or not. In the presence of censoring, the magnitude of censoring to be used and the method used to select the sample quantiles. We have derived a formula for the selection of sample quantiles for the double exponential distribution and computed the ABLUE of the quantile function and its variance. We have also analyzed in detail the factors to be considered when selecting a number of sample quantiles to represent the entire order statistics.

References

1. Ali, M.M.,D. Umbach and K.M. Hassanein(1981). Estimation of quantiles of exponential and double -exponential distributions based on two order statistics. *Commun. A-Theory/Methods* **10**, 1921-1932.
2. Cheng, S. W. (1975). A unified approach to choosing optimum quantiles for the ABLUES. *J. Amer. Statist. Assoc.* **70**, 155-159.
3. Dixon, W.J.(1957). Estimates of the mean and standard deviation of a normal population. *Ann. Math. Statist.* **28**, 806-809.
4. Dixon, W.J.(1960) Simplified estimation from censored normal samples. *Ann. Math. Statist.* **31**, 385-391.
5. Epstein, B. (1960) Estimation of life test data *Technometrics* ***2,*** 447-454
6. Hassanein, K.M (1968) An Analysis of extreme value data by sample quantiles for very large samples. *J. Amer. Statist. Assoc.* **63**, 877-888.
7. Hassanein, K.M.(1972) Simultaneous estimation of the parameters of the extreme value distribution by sample quantiles. *Technometrics* **14**, 63-70.
8. Kubat, P. and B. Epstein (1980) Estimation of quantiles of location-scale distributions based on two or three order statistics. *Technometrics* **22**, 575-582.
9. Mann, N.R. and K.W. Fertig (1977). Efficient unbiased quantile estimators for moderate size complete samples from extreme value and weibull distributions. *Technometrics* **19**, 87-93.
10. Ogawa, J. (1951). Contributions to the theory of systematic statistics I. *Osaka Math. Journal* **3**, 175-213.
11. Ogawa, J (1974). Asymptotic theory of estimation of the location and scale parameters based on selected sample quantiles- revised version. *University of Calgary, Calgary, Canada, Mimeograph report.*
12. Ogawa, J (1998).Optimal spacing of selected sample quantiles for the joint estimation of the location and scale parameter of a symmetric distribution. *Journal of statistical planning and inference* **70**, 345-360.

13. Raghunandanan, K. and Srinivasan, R. (1971). Simplified estimation of parameters in a double exponential distribution. *Technometrics* **13**, 689-702.
14. Saleh, A.K. Md. E. and Rohatgi, V.K. (1988). Estimation of parameters of location scale family of distributions in complete and type II censored samples. *Communication in Statistics-Theory and Methods* **17,** 2831-2858.
15. Sarhan, A.E. and B.G. Greenberg (1962) Simplified estimates for the exponential distribution. *Ann. Math. Statist.* **34**, 102-116.
16. Weke, P.G.O. (2005). Asymptotic Linear Estimation of the Quantile Function of a Location-Scale Family of Distributions Based on Selected Order Statistics. *The East African Journal of Statistics*, **1**,91-108.